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RAINFLOW CYCLE DISTRIBUTION AND FATIGUE DAMAGE IN GAUSSIAN RANDOM LOADINGS

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1. INTRODUCTION

Structures and mechanical components during their service life are often subjected to loads varying in a quite irregular and random manner, as those produced by wind, wave or road irregularities.

When dealing with these types of loadings, we are faced with the problem of their complete statistical characterisation, in terms of some quantities as the number of counted cycles, the distribution of their amplitudes and the total damage they cause, so to finally assess the expected fatigue life for the structure.

Fatigue assessment procedures based on cycle counting schemes and linear damage rule are capable to predict damage once the load is known, for example if measured experimentally, as in the deterministic case. However, it must kept in mind that, since an irregular load has a random nature and all computed quantities are strictly dependent on it, they are themselves random variables; for example, the set of counted cycles has to be regarded as a set of random variables, and the same holds for the amplitudes of counted cycles and their related computed damage, which is itself a random variable. From measured data we can estimate, for example, unknown parameters of an amplitude distribution with hypothesised shape (e.g.Weibull distribution), e.g. [Nagode and Fajdiga 1998]. In practice, yet, the most damaging cycles (not the most frequent) have to be considered, so statistical inference about the damage distribution of counted cycles is needed [Tovo 2000].

In any case, in order to formulate reliable statistical considerations about the distribution of counted cycles and its related fatigue damage under the linear damage rule, we need many measured load histories, which are often costly or simply time-consuming.

Moreover, in some applications as in the automotive industry, field measurements for a limited period of time serve for design purposes about the total life of the component; in this situations, there is clearly a further need to extrapolate measured data, making inference about rare events, as large cycles [Johannesson and Thomas 2001]. Consequently, sometimes experimental measurements alone cannot be viewed as reliable enough to produce satisfactory results from a statistical point of view, but require further analysis.

Then other methodologies able to reduce the time for data acquisition and analysis, and capable to guarantee a complete and reliable statistical description of the phenomenon under examination, are clearly welcome.

In the so-called spectral methods, for example, the irregular fluctuating stress is modelled as a stationary Gaussian stochastic process, described by a spectral density in the frequency domain; specifically, the irregular stress is assumed to be a broad-band process. i.e. a process with a frequency content extending over a wide range of frequencies. The first advantage of these methods lies in the possibility, once the random load under examination has been characterised by its spectral density, to simulate numerically a large number of time histories, without need to repeat different experimental measurements. The other advantage is that these spectral methods are also able to give exact or approximated analytical formulas that relate the fatigue damage under a given counting method and a given damage rule directly to the spectral density of the process. Since the rainflow count is undoubtedly the most accurate counting procedure, the main interest lies in the analysis of the statistical distribution of cycles counted by the rainflow method and in the consequent damage obtained under the linear damage rule. The more complete approach would be to relate the distribution of rainflow counted cycles directly to some characteristics of the process spectral density, since a simple analytical formula relates the fatigue damage under the linear accumulation hypothesis to the distribution of counted cycles.

The main difficulty, however, is that the complex sequential structure of cycle extraction in the rainflow algorithm makes the relationship between cycle distribution and time- (or frequency-) domain characteristics of the process very complex, and no exact closed form solutions relating the cycle distribution to the spectral density are known at present for the case of broad-band processes. Some approaches has addressed this problem either with theoretical considerations or by setting completely approximate methods, based on best fitting procedures of many simulation results; interesting results have been obtained under the Markov hypothesis for the sequence of turning points.

In other cases, simple results have been derived (often in approximate form) only for the rainflow damage under the linear rule, with only implicit or no information about the underlying statistical cycle distribution.

The first part of this paper deals with the complete characterisation of a random process both in frequency and time-domain, in terms of probability distributions (for the values of the process and its extremes, i.e. peaks and valleys) and its spectral density, with related spectral quantities (as spectral moments and bandwidth parameters).

Then, the paper faces the problem of the analysis of the distribution of counted cycles and fatigue damage by extending from the deterministic to the random case; then, it considers the most commonly used spectral methods for fatigue damage assessment under stationary Gaussian random processes, with particular attention on spectral densities with a broad-band frequency content (broad-band processes).

2. Random processes and spectral density

We assume herein that the irregular stress or strain acting in a mechanical component is modelled as a realisation $x(t)$ belonging to a stationary and ergodic random process $X(t)$, having a zero mean value. The process is uniquely characterised in time-domain by an autocorrelation function:

$$R_X(\tau) = E\left[X(t)\, X(t+\tau)\right] \tag{1}$$

where $E[\cdot]$ denotes the stochastic mean, or alternatively it is described in frequency-domain by a two-sided spectral density $S_X(\omega)$:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-i\omega\tau} d\tau \tag{2}$$

We shall often refer to a one-sided spectral density $W_X(\omega)$, defined on positive frequencies only:

$$W_X(\omega) = \begin{cases} 2\,S_X(\omega) \quad , & 0 < \omega < \infty \\ \\ S_X(0) \quad , & \omega = 0 \end{cases} \tag{3}$$

Spectral density $W_X(\omega)$ is uniquely characterised by its spectral moments:

$$\lambda_m = \int_0^{\infty} \omega^m\, W_X(\omega)\, \mathrm{d}\omega \qquad m = 1, 2, ... \tag{4}$$

which represent important time-domain properties of process $X(t)$; for example, the variance for process $X(t)$ and its derivatives $\dot{X}(t)$, $\ddot{X}(t)$ are [Lutes and Sarkani 1997]:

$$\lambda_0 = \sigma_X^2 \quad , \qquad \lambda_2 = \sigma_{\dot{X}}^2 \quad , \qquad \lambda_4 = \sigma_{\ddot{X}}^2 \tag{5}$$

Moreover, for a zero-mean Gaussian process $X(t)$, the expected rate of mean upcrossings ν^+ (i.e. crossings/sec of the mean value with positive slope) and the peak rate are, respectively [Lutes and Sarkani 1997]:

$$\nu^+ = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} \qquad ; \qquad \nu_{\mathrm{P}} = \frac{1}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}} \tag{6}$$

Spectral density $W_X(\omega)$ is also characterised by bandwidth parameters; the most used are:

$$\alpha_1 = \frac{\lambda_1}{\sqrt{\lambda_0 \lambda_2}} \quad , \quad \alpha_2 = \frac{\lambda_2}{\sqrt{\lambda_0 \lambda_4}} \tag{7}$$

which belong to a more general family of bandwidth parameters [Lutes and Sarkani 1997]:

$$\alpha_m = \frac{\lambda_m}{\sqrt{\lambda_0 \lambda_{2m}}} \tag{8}$$

Above parameters are dimensionless numbers satisfying $0 \le \alpha_1 \le 1$ and $0 \le \alpha_2 \le 1$, with $\alpha_1 \ge \alpha_2$. Furthermore, in a narrow-band process, having a spectral density centred on a restricted range of frequencies (see Figure 1(a)), α_1 and α_2 tend to unity, while for a broad-band (or wide-band) process, with a wider spectral density (see Figure 1(b)), they approach zero. Also a spectral parameter q_X, introduced by Vanmarcke [Vanmarcke 1972], is often used in alternative:

$$q_X = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}} = \sqrt{1 - \alpha_1^2} \qquad 0 \le q_X \le 1 \tag{9}$$

In a narrow-band process q_X tend to zero and in a broad-band process it approaches unity.

Other parameters can be considered, as those relative to the derivative process $\dot{X}(t)$ [Petrucci et al. 2000]:

$$\alpha_{\dot{X}} = \sqrt{\frac{\lambda_4^2}{\lambda_2 \lambda_6}} \qquad 0 \le \alpha_{\dot{X}} \le 1$$
$$q_{\dot{X}} = \sqrt{1 - \frac{\lambda_3^2}{\lambda_2 \lambda_4}} \qquad 0 \le q_{\dot{X}} \le 1 \tag{10}$$

In analogy to Eq. (8), the following set of parameters for the derivative process $\dot{X}(t)$ is defined:

$$\beta_m = \sqrt{\frac{\lambda_{m+2}^2}{\lambda_2 \lambda_{2m+2}}} \qquad 0 \le \beta_m \le 1 \tag{11}$$

It is worth noting how spectral parameters β_m involve higher order moments; from definition:

$$\beta_1 = \frac{\lambda_3}{\sqrt{\lambda_2 \lambda_4}} \quad , \quad \beta_2 = \frac{\lambda_4}{\sqrt{\lambda_2 \lambda_6}} \tag{12}$$

which have analogous meaning as α_1 and α_2 defined for $X(t)$ process.

An important time-domain characteristic of a process is the irregularity factor, IF, defined as the ratio of the mean upcrossing, ν^+, to the peak, ν_p, intensity:

$$IF = \frac{\nu^+}{\nu_p} \tag{13}$$

As α_1 and α_2 bandwidth parameters, also IF ranges from zero (broad-band processes) to unity (narrow-band processes). For Gaussian processes, it can be shown that IF equals α_2 [Lutes and Sarkani 1997].

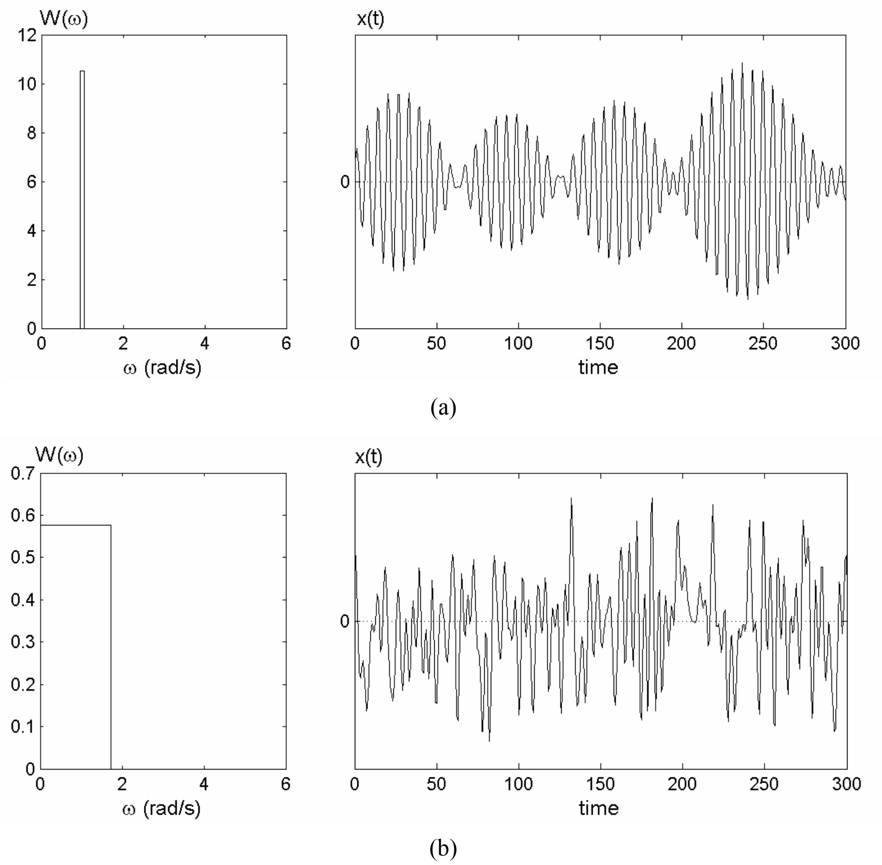


Figure 1: Examples of (a) narrow-band and (b) broad-band process.

We conclude with the probability density of peak in a Gaussian process, i.e. [Lutes and Sarkani 1997]:

$$p_p(x) = \frac{\sqrt{1-\alpha_2^2}}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{x^2}{2\sigma_X^2(1-\alpha_2^2)}} + \frac{\alpha_2\, x}{\sigma_X^2} e^{-\frac{x^2}{2\sigma_X^2}}\, \Phi\!\left(\frac{\alpha_2\, x}{\sigma_X\sqrt{1-\alpha_2^2}}\right) \tag{14}$$

which is shown in Figure 2 for different values of the α_2 parameter; its cumulative distribution is:

$$P_p(x) = \Phi\!\left(\frac{x}{\sigma_X\sqrt{1-\alpha_2^2}}\right) - \alpha_2\, e^{-\frac{u^2}{2\sigma_X^2}}\, \Phi\!\left(\frac{\alpha_2\, x}{\sigma_X\sqrt{1-\alpha_2^2}}\right) \tag{15}$$

being $\Phi(-)$ the standard normal distribution function.

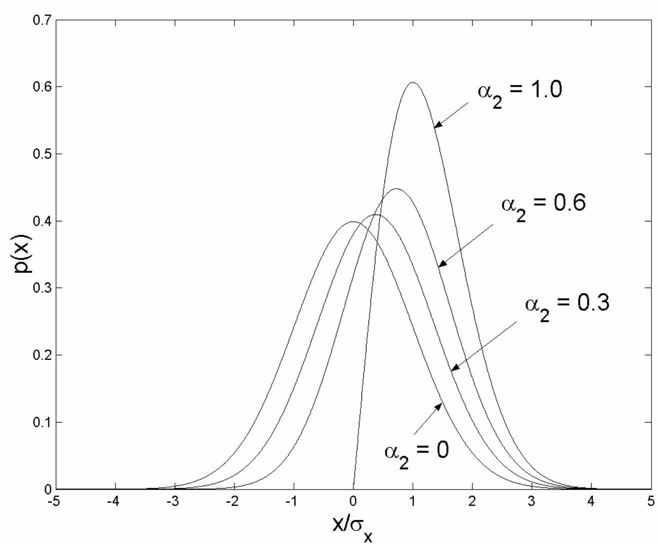


Figure 2: Probability density of peaks, $p_{\mathrm{p}}(x)$, for different values of α_2 parameter.

The density function of valleys is symmetrical with that of peaks, $p_{\mathrm{v}}(x) = p_{\mathrm{p}}(-x)$. For narrow-band processes ($\alpha_1 = \alpha_2 = 1$), the peak distribution turns into a Rayleigh distribution; for processes with mean value different from zero, equation above can be easily updated by means of a variable shift [Lutes and Sarkani 1997].

3. Properties of the distribution of counted cycles

Fatigue damage is related to amplitudes and mean values of stress cycles; for a variable amplitude load, a given counting method extracts cycles by pairing peaks and valleys in the load. The set of counted cycles obviously depends on the load examined, thus, if the load represents a time history $x(t)$ belonging to a random process $X(t)$, it has to be regarded as a set of random variables.

The fundamental problem of the fatigue assessment framework is to find, for a random process $X(t)$, the true distribution of counted cycles under a chosen counting method, e.g. the rainflow count. More precisely, the aim is to establish the correlation between the spectral density of the process and the probability density of its cycles counted by the rainflow method, since fatigue damage under the linear damage rule actually depends on this cycle distribution.

Thus, the distribution of rainflow counted cycles plays a fundamental role in the entire fatigue assessment procedure; however, the explicit correlation between this distribution and the spectral density of a random process is not known.

The statistical description of cycle distribution will be addressed in the following sections by the introduction of two alternative (but related) descriptors, namely the count intensity (introduced by [Rychlik 1993b]) and the joint density function, and by the definition of their general properties.

Let $x(t)$, $0 \le t \le T$, a time history taken from random process $X(t)$. Let us suppose that a counting method gives a finite set of $N(T)$ counted cycles $\left\{ (m_k^*, M_k) \right\}$, where M_k and m_k^* are the maximum and the minimum of each cycle, with $m_k^* < M_k$, and where can be for example $m_k^* = m_k^{\mathrm{rfc}}$ for the rainflow count or $m_k^* = m_k^{\mathrm{rc}}$ for the range (max-min) count. Let $N_T(u, \mathrm{v})$ be the number of cycles counted in $x(t)$ such that the maximum M_k is higher than u and the attached minimum m_k^* is lower than v, i.e.:

$$N_T(u, \mathrm{v}) = \#\left\{ (m_k^*, M_k) \ : \ M_k > u \ge \mathrm{v} > m_k^* \right\} \tag{16}$$

where $\#\{\cdot\}$ is the number of elements in the set $\{\cdot\}$. Function $N_T(u,\mathrm{v})$ is called the count distribution. Further, let $\mu_T(u,\mathrm{v})$ denote the expected value of $N_T(u,\mathrm{v})$, and define the count intensity as:

$$\mu(u,\mathrm{v}) = \lim_{T\to\infty} \frac{\mu_T(u,\mathrm{v})}{T} \qquad \text{with} \qquad \mu_T(u,\mathrm{v}) = E\left[\, N_T(u,\mathrm{v}) \,\right] \tag{17}$$

assuming the limit exists (for ergodic processes the limit exists). Note that for stationary loads:

$$\mu_T(u,\mathrm{v}) = T\ \mu(u,\mathrm{v}) \tag{18}$$

The expected count $\mu_T(u,\mathrm{v})$ can be thought as the asymptotic shape of the counting distribution $N_T(u,\mathrm{v})$ and it retains all properties of the statistical distribution of cycles counted in process $X(t)$.

Note that $\mu_T(u,\mathrm{v})$ is an increasing function of v and a decreasing function of u, i.e. for any fixed $u \geq \mathrm{v}$:

$$\mu_T(u,\mathrm{v}) \ \leq\ \min\left(\, \mu_T(u,u), \mu_T(\mathrm{v},\mathrm{v}) \,\right) = \ \mu_T^{+}(u,\mathrm{v}) \tag{19}$$

Thus, the count intensity is bounded by its diagonal values. This property will be subsequently used to construct an upper bound for fatigue damage.

According to previous definition, $\mu_T(u,u)$ is the number of cycles (m_k^*, M_k) such that $M_k > u > m_k^*$. Let us define a cycle count as a crossing-consistent method if $\mu_T(u,u)$ equals the expected number of *u*-upcrossings of process $X(t)$, i.e.:

$$\mu_T(u,u) = \#\{\ t_k :\ t_k \text{ is a } u\text{-upcrossing of } x(t)\ \} \tag{20}$$

For simplicity of notation, write $\mu_T(u,u) = \mu_T(u)$; for Gaussian processes, an explicit expression for the number of upcrossings in time T is given by Rice's formula [Lutes and Sarkani 1997]:

$$\mu_T(u) = T \frac{1}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}}\ e^{-\frac{u^2}{2\lambda_0}} \tag{21}$$

Let us also define a counting method as a complete counting if it pairs every peak in a load with a lower or equal valley, so that the number of all counted cycles is equal to the total number of peaks. Note that a complete counting method is also crossing-consistent [Tovo 2002].

Amongst usual counting methods, the range (min-max) count is both complete and crossing-consistent, the level-crossing is clearly crossing-consistent, whereas the peak-valley count is neither complete nor crossing-consistent. The rainflow count needs instead some cautions, since it can be defined according to different algorithms, and residuals (i.e. unclosed loops) often occur in several definitions; however, modifications can be adopted to count only full cycles (e.g. case of repeating histories); thus, the rainflow count is actually a complete method.

We shall introduce now an alternative (and complementary) descriptor of the statistical distribution of counted cycles based on probability density concept.

Let $h(u,\mathrm{v})$ be the joint distribution of cycles counted in random process $X(t)$, as a function of peak u and valley v levels; note that $h(u,\mathrm{v})$ is null for $u < \mathrm{v}$. The related cumulative distribution function:

$$H(u,\mathrm{v}) = \int_{-\infty}^{u} \int_{-\infty}^{\mathrm{v}} h(x,y)\, \mathrm{d}x\, \mathrm{d}y \tag{22}$$

gives the probability to count a cycle with peak lower or equal to level u and valley lower or equal to level v. By a simple change of variables, we can express the cycle distribution in terms of amplitudes and mean values:

$$p_{\mathrm{a,m}}(s,m)=2\ h(m+s,m-s) \tag{23}$$

while the distribution of counted cycles as a function of amplitudes is given by the marginal probability density function:

$$p_{\mathrm{a}}(s)=\int_{-\infty}^{+\infty} p_{\mathrm{a,m}}(s,m)\,\mathrm{d}m \tag{24}$$

In a complete counting method, a cycle is attached to each maximum in the history, thus the expected intensity of counted cycles, say ν_{a}, is clearly equal to the expected intensity of peaks, i.e. $\nu_{\mathrm{a}}=\nu_{\mathrm{p}}$. Similarly, since due to completeness every peak is paired with an equal or lower valley, the marginal distribution of cycles must be also related to the distributions of peaks and valleys [Tovo 2002]:

$$\begin{cases} p_{\mathrm{p}}(u)=\displaystyle\int_{-\infty}^{u} h(u,\mathrm{v})\ \mathrm{dv} \\ \\ p_{\mathrm{v}}(\mathrm{v})=\displaystyle\int_{\mathrm{v}}^{+\infty} h(u,\mathrm{v})\ \mathrm{d}u \end{cases} \tag{25}$$

This relation is very general and it holds for both Gaussian and non-Gaussian processes, when proper peak and valley distribution are considered.

Every counting method defines its own probability density of counted cycles, $h(u,\mathrm{v})$, but, if the method is complete, this distribution must satisfy Eq. (25); thus, also the distribution associated to the rainflow count (which is a complete procedure) is a solution of the above equations. In other words, these equations stand as a necessary condition for the cycle distribution from any complete counting method; moreover, since the equations are linear, either any linear combination of two solutions will be a solution. This condition will be used in subsequent sections to construct a method for estimating the distribution of rainflow cycles.

For a given counting method, both $H(u,\mathrm{v})$ and $\mu(u,\mathrm{v})$ functions are two alternative cumulative distributions for the same set of counted cycles. On the basis of the definition of the $\mu(u,\mathrm{v})$ function, we have that:

$$\mu(u,\mathrm{v})=\nu_{\mathrm{a}}\int_{x=u}^{+\infty}\int_{-\infty}^{y=\mathrm{v}} h(x,y)\ \mathrm{d}x\ \mathrm{d}y \tag{26}$$

where for complete counts we can put $\nu_{\mathrm{a}}=\nu_{\mathrm{p}}$.

We conclude with another important descriptor of the cycle distribution usual in engineering practice, namely the cumulative (or loading) cycle spectrum, defined as the percentile number of cycles having amplitudes higher than or equal to s, i.e.:

$$F(s)=\int_{s}^{+\infty} p_{a}(x)\ \mathrm{d}x \tag{27}$$

where $p_{\mathrm{a}}(s)$ is the probability density of the amplitude of counted cycles.

4. FATIGUE DAMAGE

Knowledge of the statistical distribution of cycles counted in random process $X(t)$ allows one to calculate total fatigue damage under the Palmgren-Miner rule, once constant amplitude fatigue properties are given (as the S-N curve). For instance, we shall compute fatigue damage based either on the marginal amplitude distribution $p_{\mathrm{a}}(s)$ or on the expected count $\mu_T(u,\mathrm{v})$.

In time history $x(t)$, $0 \le t \le T$, total damage under the linear damage rule is:

$$D(T) = \sum_{i=1}^{N(T)} \Delta D_i = \sum_{i=1}^{N(T)} \frac{1}{N_i} \tag{28}$$

being ΔD_i the damage increment associated to each *i*-th counted cycle, N_i the number of cycles to failure associated to stress amplitude s_i, and $N(T)$ the number of all counted cycles. Since damage increment ΔD_i depends on constant amplitude fatigue properties through the S-N curve, previous formula becomes:

$$D(T) = \sum_{i=1}^{N} \frac{s_i^k}{C} \tag{29}$$

where s_i is the amplitude of the *i*-th counted cycle and $s^k N = C$ is the S-N curve. In a random process, s_i is obviously a random variable, whose distribution clearly depends on process itself and on the counting method used (e.g. rainflow count), thus total damage $D(T)$ is a random variable too (precisely, damage $D(t)$ as a function of time variable t is a non-decreasing random process, otherwise it may be more correctly defined as a rate-independent functional defined on $X(t)$).

It is very difficult in general to find the exact distribution of $D(T)$, also in the validity of linear damage accumulation hypothesis. This because, even if the probabilistic structure of process $X(t)$ is well defined, damage is a complicate non-linear functional defined on $X(t)$. Consequently, we shall concentrate mainly on expected quantities for fatigue life and damage, this one evaluated under the linear damage accumulation rule. For instance, some approximations adopt a normal distribution for damage [Kececioglu et al. 1998], often under the Markov assumption for the sequence of extremes [Rychlik et al. 1995]; in other cases, as under non-linear damage accumulation rules, more complex theoretical frameworks can be developed [Rejman and Rychlik 1993].

If the number of counted cycles is large, and in the hypothesis that amplitudes have same distribution and that dependence between them is weak (i.e. amplitudes are assumed to be independent and identically distributed), the expected total damage value for the process, calculated by taking expectation of Eq. (29), is [Madsen et al. 1986]:

$$\overline{D}(T) = E[D(T)] = E\left(\sum_{i=1}^{N_T} \frac{s_i^k}{C} \right) = E[N_T] \frac{E[s^k]}{C} \tag{30}$$

writing $\overline{N} = E[N(T)]$ for the expected number of cycles counted in time T; this result is valid independent of the counting method.

In stationary processes, $\overline{N} = \nu_{\mathrm{a}} T$, where ν_{a} is the expected intensity of counted cycles; in complete counts (as the rainflow method), $\nu_{\mathrm{a}} = \nu_{\mathrm{p}}$, where ν_{p} uniquely depends on the spectral density of process $X(t)$ (see Eq. (6)). Furthermore, in stationary processes we have that $\overline{D}(T) = T\,\overline{D}(1)$, where $\overline{D}(1)$ is called the expected damage intensity (i.e. the damage per time unit).

Equation (30) establishes that the expected damage (or, equivalently, the damage intensity), is related to the k-th moment of the amplitude distribution, $p_{\mathrm{a}}(s)$. By neglecting mean values effect, for sake of simplicity, the explicit formula for calculating the expected damage intensity is:

$$\overline{D}(1) = \nu_a \, C^{-1} \int_0^{+\infty} s^k \, p_a(s) \, ds \tag{31}$$

In the following, we shall often write $\overline{D}(1) = \overline{D}$ for the damage intensity.

Formula above clearly states that, for a given random process $X(t)$ (i.e. for a given spectral density), damage intensity uniquely depends on the counting method adopted though the distribution $p_{\mathrm{a}}(s)$, or equivalently $h(u, \mathrm{v})$. We shall refer in particular to the rainflow counting method, which amongst all algorithms has been recognised as the best one [Dowling 1972]; then, distribution $h_{\mathrm{RFC}}(u, \mathrm{v})$ of rainflow cycles will give the rainflow fatigue damage intensity under the linear rule, $\overline{D}_{\mathrm{RFC}}$, through its marginal distribution $p_{\mathrm{a}}^{\mathrm{RFC}}(s)$ substituted into Eq. (31).

Evidently rainflow distribution $h_{\mathrm{RFC}}(u, \mathrm{v})$ plays a fundamental role in the entire fatigue damage assessment procedure; unfortunately, because of the complicate sequential structure on which the rainflow method is based, at present no explicit analytical solution is available for $h_{\mathrm{RFC}}(u, \mathrm{v})$ density, so that closed-form expressions for rainflow damage are not available.

Damage $D(T)$ may be alternatively related to the cycle distribution properties of process $X(t)$ by the use of the counting distribution $N_T(u, \mathrm{v})$, as suggested by Rychlik [Rychlik 1993b].

Let us consider a cycle (u, v) having maximum and minimum at levels u and v, respectively, and denote by $g(u, \mathrm{v})$ the damage it causes, according to the S-N curve, i.e. $g(u - \mathrm{v}) = C^{-1}(u - \mathrm{v})^k$. In the hypothesis that $g(0) = 0$ and that $N_T(u, u)$ is a bounded function of u, an integration by part argument shows that total damage $D(T)$, under the linear damage rule, is finite and given by [Frendhal and Rychlik 1993]:

$$D(T) = g'(0) \int_{-\infty}^{+\infty} N_T(u, u) \, \mathrm{d}u + \int_{-\infty}^{+\infty} \int_{-\infty}^{u} N_T(u, \mathrm{v}) \, g''(u - \mathrm{v}) \, \mathrm{dv} \, \mathrm{d}u \tag{32}$$

This integral damage formulation links total damage to the cumulative count distribution $N_T(u, \mathrm{v})$; by replacing the counting distribution $N_T(u, \mathrm{v})$ by its expectation $\mu_T(u, \mathrm{v})$, we shall obtain the formula for the expected damage $\overline{D}(T)$ [Rychlik 1993b], i.e.:

$$\overline{D}(T) = g'(0) \int_{-\infty}^{+\infty} \mu_T(u, u) \, \mathrm{d}u + \int_{-\infty}^{+\infty} \int_{-\infty}^{u} \mu_T(u, \mathrm{v}) \, g''(u - \mathrm{v}) \, \mathrm{dv} \, \mathrm{d}u \tag{33}$$

In other words, while the counting distribution $N_T(u, \mathrm{v})$ defines the total damage, $D(T)$, the expected count $\mu_T(u, \mathrm{v})$ defines the expected damage, $\overline{D}(T)$, and correspondingly the counting intensity $\mu(u, \mathrm{v})$ will give the expected damage intensity, $\overline{D}$.

Furthermore, if $g''(s) \geq 0$ for all amplitudes $s = u - \mathrm{v} \geq 0$, one can use Eq. (33) to compare damages from different counting procedures, based on their distribution properties.

Let us consider in particular the range and the rainflow counts. On the basis of their algorithms, given maximum M_k of a generic counted cycle, the attached rainflow counted minimum m_k^{rfc} is always equal or lower than the attached minimum counted by the range-count, m_k^{rc}; namely, for a given maximum M_k, it is always

$m_k^{\mathrm{rfc}} \le m_k^{\mathrm{rc}}$, being m_k^{rfc} the rainflow minimum, and m_k^{rc} the range-count minimum, from which it follows that $\mu_T^{\mathrm{rc}}(u,\mathrm{v}) \;\le\; \mu_T^{\mathrm{rfc}}(u,\mathrm{v})$. In addition, as stated by Eq. (19), the counting distribution for a crossing-consistent method is always bounded by its diagonal values, thus we can write:

$$\mu_T^{\mathrm{rc}}(u,\mathrm{v}) \;\le\; \mu_T^{\mathrm{rfc}}(u,\mathrm{v}) \;\le\; \min\big(\, \mu_T(u,u),\, \mu_T(\mathrm{v},\mathrm{v}) \,\big) \tag{34}$$

Based on Eqs. (33) and (34), we conclude that damage using rainflow count always bounds the damage obtained using the range count; furthermore, an upper bound, say $D^+(T)$, for all crossing-consistent counting methods exists, i.e.:

$$D_{\mathrm{RC}}(T) \;\le\; D_{\mathrm{RFC}}(T) \;\le\; D^+(T) \tag{35}$$

Equation above is true for a given load defined in time interval $[0,T]$, and it equivalently holds for damage intensities as well. We shall see later on that the upper bound coincides for Gaussian loads with the narrow-band approximation.

In conclusion, we can affirm that the expected rainflow damage intensity, i.e. the main quantity to be investigated, strictly depends on the distribution of rainflow counted cycles, $h_{\mathrm{RFC}}(u,\mathrm{v})$. Unfortunately, because of the complicate sequential structure on which the rainflow method is based, at present no explicit analytical solution is available for the $h_{\mathrm{RFC}}(u,\mathrm{v})$ density, and so for the related expected damage.

5. Analytical solution for fatigue damage

The main problem in the fatigue damage assessment procedure is the estimation of the expected rainflow damage intensity $\overline{D}_{\mathrm{RFC}}$ for random process $X(t)$ and its related fatigue life. As evidenced by previous discussion, this problem can be solved by first estimating the true rainflow cycle distribution $h_{\mathrm{RFC}}(u,\mathrm{v})$ and then computing damage intensity $\overline{D}_{\mathrm{RFC}}$ under the linear damage hypothesis; alternatively, direct estimation of fatigue damage is possible as well.

Methods addressing this problem can be divided essentially into few categories: some of them first estimate the true rainflow cycle distribution (as the joint density, $h_{\mathrm{RFC}}(u,\mathrm{v})$, or its marginal density, $p_{\mathrm{a}}(s)$) and then compute damage under the linear rule according to Eq. (31) [Dirlik 1985, Zhao and Baker 1992, Tovo 2002]; other methods, instead, give exact or approximate formulas for directly estimating rainflow damage $\overline{D}_{\mathrm{RFC}}$, without information about the underlying cycle distribution [Wirsching and Light 1980]; finally, other methods estimate the rainflow damage by adopting the Markov hypothesis for the sequence of extremes [Frendhal and Rychlik 1993].

A further problem is also to establish the dependence existing between the rainflow cycle distribution (or rainflow damage) and some frequency-domain characteristics of process $X(t)$, namely its spectral density, and specifically to investigate the true set of bandwidth parameters involved in this dependence. In the next we shall give a brief review of methods applicable to stationary Gaussian random processes; a complete survey can be found in [Bouyssy et al. 1993].

5.1. Peak approximation

Damage in process $X(t)$ can be estimated under the peak-valley counting assumption, in which each peak level determines the corresponding cycle amplitude; the amplitude distribution is then estimated according to the peak distribution and damage intensity becomes [Tovo 2002]:

$$\overline{D}_{\mathrm{PV}} = \nu_{\mathrm{p}}\, C^{-1} \int_0^{+\infty} s^k\, p_{\mathrm{p}}(s)\, ds \tag{36}$$

For Gaussian processes, the peak distribution $p_{\mathrm{p}}(s)$ is given by Eq. (14) and consequently the damage intensity is given by two contributions:

$$\overline{D}_{\mathrm{PV}} = \nu_{\mathrm{p}}\, C^{-1} \left[\int_0^{+\infty} s^k \frac{\sqrt{1-\alpha_2^2}}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{s^2}{2\sigma_X^2(1-\alpha_2^2)}} ds \; + \right.$$
$$\left. + \int_0^{+\infty} s^k \frac{\alpha_2\, s}{\sigma_X^2} e^{-\frac{s^2}{2\sigma_X^2}}\, \Phi\!\left(\frac{\alpha_2\, s}{\sigma_X \sqrt{1-\alpha_2^2}} \right) ds \right] \tag{37}$$

Integration limits clearly consider only contribution from positive peaks. This approach has been called the peak approximation of fatigue damage [Lutes and Sarkani 1997].

It is worth noting that the peak-valley counting is not a complete procedure, in fact in calculating fatigue damage in broad-band loadings it completely neglects the fraction of negative peaks. In addition, as in the deterministic case, the peak-valley count generally overestimates total damage (particularly for irregular loadings), being greater even than damage from the narrow-band approximation, as evidenced in [Tovo 2002].

On the basis of these considerations, although Lutes and Sarkani support the peak approximation [Lutes and Sarkani 1997], in our opinion is not useful to estimate fatigue damage through the integral deriving from the peak-count as reported in Eq. (37), as done in [Lu et al. 1998], since this approach has been shown to give a damage predictor always above the upper damage value estimated by any complete and crossing consistent counting method (see [Tovo 2002]).

We note that another simple approach approximating the rainflow amplitude distribution through the peak distribution is proposed in literature [Kim & Kim 1994], even if it produces not satisfactory results (see [Petrucci and Zuccarello 1999]).

5.2. Narrow-band approximation

For a strictly narrow-band Gaussian process $X(t)$, as that depicted in Figure 1(a), it is reasonable to assume the amplitude distribution $p_{\mathrm{a}}(s)$ coincident with the peak distribution $p_{\mathrm{p}}(x)$, which in a narrow-band process is Rayleigh; furthermore, the intensity of counted cycles, ν_{a}, can be taken equal to the mean upcrossing intensity, ν^+, given by Eq. (6). Accordingly, calculating the fatigue damage intensity as in Eq. (31), one finds:

$$\overline{D}_{\mathrm{NB}} = \nu^+\, C^{-1} \left(\sqrt{2\,\lambda_0}\right)^k\, \Gamma\!\left(1+\frac{k}{2}\right) \tag{38}$$

where $\Gamma(-)$ is the Gamma function. Equation above is valid for a S-N curve with a single slope over the whole range of amplitudes; a closed-form solution including a slope change is developed in [Tunna 1986].

Previous expression holds exactly only for a strictly narrow-band process $X(t)$. When instead it is applied to a process $X(t)$ being broad-band, the predicted damage value is that of an equivalent ideal narrow-process, with the same variance and a number of peaks equal to the number of upcrossings (or downcrossings) of the mean level of real broad-band process $X(t)$. This is the so-called narrow-band (or Rayleigh) approximation of the fatigue damage of a broad-band process and has frequently been used in engineering applications.

It is widely accepted the fact that the narrow-band approximation, when applied to wide-band processes, tends to over-estimate the rainflow fatigue damage, and that was proved rigorously by Rychlik [Rychlik 1993a].

Hence, some authors proposed to approximate the rainflow damage intensity $\overline{D}_{\mathrm{RFC}}$ by correcting (namely, by reducing) the damage value predicted by the narrow-band approximation [Wirsching and Light 1980]:

$$\overline{D}_{\mathrm{RFC}} = \lambda_{\mathrm{WL}}\ \overline{D}_{\mathrm{NB}} \tag{39}$$

in which λ_{WL} is an empirical correction factor assumed to be dependent on the fatigue curve parameters and on α_2 bandwidth parameter:

$$\lambda_{\mathrm{WL}} = a(k) + \left[\, 1 - a(k)\, \right](1-\varepsilon)^{b(k)} \tag{40}$$

where $\varepsilon = \sqrt{1-\alpha_2^2}$ is a spectral width parameter and $\alpha(k)$ and $b(k)$ are best fitting parameters expressed as:

$$a(k) = 0.926 - 0.033k \quad ; \qquad b(k) = 1.587k - 2.323 \tag{41}$$

The particular form was established as being reasonable on the basis of observation of the data obtained from a rainflow analysis of simulated samples of some broad-band processes, and it is quite simple.

For a narrow-band process, $\alpha_2 = 1$ ($\varepsilon = 0$), which gives correctly $\lambda_{\mathrm{WL}} = 1$. We note also that previous formula assumes the rainflow damage to be dependent on just three spectral moments (i.e. λ_0, λ_2 and λ_4), through α_2 parameter.

This approach find general application in practical problems concerning wind and wave induced random loadings [Siddiqui and Ahmad 2001, Holmes 2002].

5.3. Approximation for the rainflow amplitude distribution

Some methods presented in literature try to directly estimate the rainflow amplitude distribution $p_{\mathrm{RFC}}(s)$, from which to compute fatigue damage under the linear rule according to Eq. (31).

The crucial problem is that in general we don't know the true shape of the rainflow amplitude distribution, and also what set of spectral parameters actually relate this distribution to the process spectral density. Consequently, some kind of parametric shape must be assumed in advance and then calibrated through a best fitting procedure over extensive numerical simulations or experimental data. For instance, densities used are often of Rayleigh, Exponential or Weibull type, or some kind of mixture [Wirsching and Sheata 1977, Bouyssy et al. 1993].

The advantage of the knowledge of the amplitude distribution is twofold: first of all, it still allows estimation of fatigue damage under the linear rule by simple integration as in Eq. (31); furthermore, it allows treatment of rare events, as large cycles, by extrapolation of the amplitude distribution towards large amplitude values.

5.3.1. Dirlik approximate model (1985)

Probably the most famous empirical formula for approximating the rainflow amplitude distribution is that proposed by Dirlik [Dirlik 1985], which uses a combination of an Exponential and two Rayleigh densities.

In the Dirlik's model, the approximate closed-form expression for the probability density function of rainflow ranges r is:

$$p_{\mathrm{RFC}}^{\mathrm{Dir}}(r)=\frac{1}{2(\lambda_0)^{1/2}}\left[\frac{D_1}{Q}e^{-\frac{Z}{Q}}+\frac{D_2 Z}{R^2}e^{-\frac{Z^2}{2R^2}}+D_3\,Ze^{-\frac{Z^2}{2}}\right] \tag{42}$$

where:

$$Z=\frac{r}{2\,(\lambda_0)^{1/2}}=\frac{s}{(\lambda_0)^{1/2}} \tag{43}$$

is the normalized amplitude and:

$$x_m=\frac{\lambda_1}{\lambda_0}\left(\frac{\lambda_2}{\lambda_4}\right)^{1/2} \quad , \qquad D_1=\frac{2\,(x_m-\alpha_2^2)}{1+\alpha_2^2}$$

$$D_2=\frac{1-\alpha_2-D_1+D_1^2}{1-R} \quad , \quad D_3=1-D_1-D_2 \tag{44}$$

$$Q=\frac{1.25(\alpha_2-D_3-(D_2R))}{D_1} \quad , \quad R=\frac{\alpha_2-x_m-D_1^2}{1-\alpha_2-D_1+D_1^2}$$

are parameters resulting from a best fitting procedure over a large set of data from numerical simulations. It can be easily verified that $x_m=\alpha_1\cdot\alpha_2$ and that $D_1/Q=0.8$ (i.e. first coefficient in the distribution is constant).

The amplitude probability density, say $p_{\mathrm{RFC}}^{\mathrm{Dir}}(s)$, follows from a simple variable transformation:

$$p_{\mathrm{RFC}}^{\mathrm{Dir}}(s)=\frac{1}{(\lambda_0)^{1/2}}\left[\frac{D_1}{Q}e^{-\frac{Z}{Q}}+\frac{D_2 Z}{R^2}e^{-\frac{Z^2}{2R^2}}+D_3\,Z\,e^{-\frac{Z^2}{2}}\right] \tag{45}$$

being as usual Z the normalized amplitude. It should be noted that, if compared to the Wirsching and Light' model, this approach gives an amplitude distribution (and thus a rainflow damage) depending on just four spectral moments (i.e. λ_0, λ_1, λ_2 and λ_4), including in particular a dependence on λ_1 moment.

The rainflow damage intensity under the Palmgren-Miner rule is calculated by substituting $p_{\mathrm{RFC}}^{\mathrm{Dir}}(s)$ density in Eq. (31):

$$\overline{D}_{\mathrm{RFC}}^{\mathrm{Dir}}=\frac{\nu_{\mathrm{p}}}{C}\,\lambda_0^{k/2}\left[D_1\,Q^k\,\Gamma(1+k)+\left(\sqrt{2}\right)^k\,\Gamma\!\left(1+\frac{k}{2}\right)\left(D_2\,|R|^k+D_3\right)\right] \tag{46}$$

Many works have evidenced how Dirlik's formula is far superior to other existing methods in estimating rainflow fatigue damage [Bouyssy et al. 1993, Halfpeny 1999].

However, we can notice how Dirlik's method has some drawbacks. First of all, it is proposed as a completely approximate approach, not supported by any kind of theoretical framework; secondly, the proposed rainflow distribution does not account for mean value dependence, making so impossible a further extension to cover also non-Gaussian problems.

Finally, we compute the loading spectrum as in Eq. (27):

$$F^{\text{Dir}}(s) = D_1\, e^{-\frac{Z}{Q}} + D_2\, e^{-\frac{Z^2}{2R^2}} + D_3\, e^{-\frac{Z^2}{2}} \tag{47}$$

which, according to the definition of parameters, satisfies $F(s=0)=1$.

5.3.2. Zhao and Baker model (1992)

Zhao and Baker used a similar concept, by assuming that amplitude probability distribution is a linear combination of one Weibull and one Rayleigh density [Zhao and Baker 1992]:

$$p_{\text{RFC}}^{\text{ZB}}(Z) = w\;\, a\, b\, Z^{b-1}\, e^{-a Z^b} + (1-w)\;\, Z\, e^{-\frac{Z^2}{2}} \tag{48}$$

where Z is the normalised amplitude defined in Eq. (43), w is a weighting factor ($0 \le w \le 1$), and a, b are the Weibull parameters ($a>0$, $b>0$). Previous parameters, depending on spectral properties of process $X(t)$, are determined from simulations on a wide range of spectra, but are also supported by some theoretical arguments. Specifically, the weighting factor is:

$$w = \frac{1-\alpha_2}{1-\sqrt{\frac{2}{\pi}}\,\Gamma\left(1+\frac{1}{b}\right) a^{-1/b}} \tag{49}$$

while the other two parameters are:

$$a = 8 - 7\,\alpha_2 \quad , \quad b = \begin{cases} 1.1 & \text{if } \alpha_2 < 0.9 \\ 1.1 + 9(\alpha_2 - 0.9) & \text{if } \alpha_2 \ge 0.9 \end{cases} \tag{50}$$

For a narrow-band process, $\alpha_2 = 1$, which gives $a=1$, $b=2$ and $w=0$, implying for the amplitudes a Rayleigh distribution, which is the exact distribution. However, according to the definition of a and b parameter given above, when $\alpha_2 \le 0.130$ it happens that $w>1$, which is not correct; however, applications having so small values of α_2 are not so frequent in practice.

An alternative improved version of a parameter, which includes an additional functional relationship on $\alpha_{0.75} = \lambda_{0.75} \big/ \sqrt{\lambda_0\, \lambda_{1.5}}$ bandwidth parameter, exists. In fact, it was observed by simulations that, for small values of k (e.g. $k=3$), rainflow damage is more closely correlated with other spectral properties than with α_2 [Lutes at. al 1984]. Specifically, the correction factor $\lambda = \overline{D}_{\text{RFC}} \big/ \overline{D}_{\text{NB}}$ defined as for example in Eq. (39) has been correlated with $\alpha_{0.75}$, for $k=3$, by the following formula [Zhao and Baker 1992]:

$$\lambda_{ZB}\big|_{k=3} = \begin{cases} -0.4154 + 1.392\,\alpha_{0.75} & \text{if } \alpha_{0.75} \ge 0.5 \\ 0.28 & \text{if } \alpha_{0.75} < 0.5 \end{cases} \tag{51}$$

Then, a is calculated as $a = d^{-b}$, being d a solution of:

$$\Gamma\left(1+\frac{3}{b}\right)(1-\alpha_2)d^3 + 3\Gamma\left(1+\frac{1}{b}\right)(\lambda_{\text{ZB}}\alpha_2 - 1)\,d + 3\sqrt{\frac{\pi}{2}}\,\alpha_2\,(1-\lambda_{\text{ZB}}) = 0 \tag{52}$$

In the case of a narrow-band process, $\alpha_{0.75} = 1$, so giving $\lambda_{\mathrm{ZB}} = 0.9766$, which is not coincident with the exact solution being expected (i.e. $\lambda_{\mathrm{ZB}} = 1$); furthermore, by adopting this alternative definition, it can happen that $w < 0$ when considering particular values of $\alpha_{0.75}$ and α_2 (e.g. $\alpha_2 > 0.5$ and $\alpha_{0.75} < 0.65$).

All details for calculating all parameters can be found in [Zhao and Baker 1992].

The amplitude density defined in Eq. (48) depends on the normalised amplitude; by a simple variable change, we can express it function of amplitude s, i.e.:

$$p_{\mathrm{RFC}}^{\mathrm{ZB}}(s) = w \; \frac{a\,b}{\lambda_0^{1/2}} \left(\frac{s}{\lambda_0^{1/2}} \right)^{b-1} e^{-a\left(\frac{s}{\lambda_0^{1/2}}\right)^{b}} + (1-w) \; \frac{s}{\lambda_0} \, e^{-\frac{1}{2}\left(\frac{s}{\lambda_0^{1/2}}\right)^{2}} \tag{53}$$

The rainflow damage intensity under the Palmgren-Miner rule is calculated by substituting $p_{\mathrm{RFC}}^{\mathrm{ZB}}(s)$ density as in Eq. (31):

$$\overline{D}_{\mathrm{RFC}}^{\mathrm{ZB}} = \frac{\nu_{\mathrm{p}}}{\mathrm{C}} \lambda_0^{k/2} \left[w\, \alpha^{-\frac{k}{\beta}} \, \Gamma\!\left(1+\frac{k}{\beta}\right) + (1-w)\; 2^{\frac{k}{2}} \, \Gamma\!\left(1+\frac{k}{2}\right) \right] \tag{54}$$

Finally, we give the loading spectrum:

$$F^{\mathrm{ZB}}(s) = w\, e^{-a\left(\frac{s}{\lambda_0^{1/2}}\right)^{b}} + (1-w)\, e^{-\frac{s}{2\,\lambda_0^{1/2}}} \tag{55}$$

5.3.3. Benasciutti and Tovo method (2002)

In this section we shall analyse in detail an alternative method for estimating the rainflow cycle distribution (and the related fatigue damage under the linear rule) in a Gaussian broad-band processes.

Let $x(t)$, $0 \le t \le T$ be a time history belonging to a Gaussian random process $X(t)$. Now, let us rewrite Eq. (35) in terms of damage intensities:

$$\overline{D}_{\mathrm{RC}} \;\le\; \overline{D}_{\mathrm{RFC}} \;\le\; \overline{D}^{+} \;=\; \overline{D}_{\mathrm{NB}} \tag{56}$$

This equation states that, under the linear damage accumulation rule, the rainflow damage always bounds the damage from the range-count, and that an upper damage value, $\overline{D}^{+}$, exists, which bounds damage computed by any crossing-consistent counting method (see sections 3 and 4).

Rychlik has proved that for Gaussian processes, the upper bound coincides with the damage given by the narrow-band approximation, as reported in Eq. (56) [Rychlik 1993a]. Furthermore, it has been recently pointed out that the narrow-band damage also equals the damage calculated under the linear damage rule for the level-crossing count [Tovo 2002].

Precisely, in the level-crossing count all positive peaks, reduced by the number of positive valleys at the same level, are paired with the lowest available valley, which is symmetric in a symmetric process, to form damaging cycles with non-zero amplitude. The remaining peaks and valleys at same level are paired together to form zero-amplitude (non-damaging) cycles. This leads to the following distribution of level-crossing counted cycles as a function of peak and valley levels:

$$h_{\mathrm{LCC}}(u,\mathrm{v}) = \tag{57}$$
$$= \begin{cases} \left[p_{\mathrm{p}}(u) - p_{\mathrm{v}}(u) \right] \delta(u+\mathrm{v}) \; + \; p_{\mathrm{v}}(u)\, \delta(u-\mathrm{v}) & \text{if } \; u>0 \\ p_{\mathrm{p}}(u)\, \delta(u-\mathrm{v}) & \text{if } \; u \le 0 \end{cases}$$

where δ is the Dirac delta function and $p_{\mathrm{p}}(u)$ and $p_{\mathrm{v}}(u)$ are the peak and valley distributions, respectively. The component related to $\delta(u-\mathrm{v})$, represents zero-amplitude cycles, has no damaging effect and may be neglected in practical applications; the component related to $\delta(u+\mathrm{v})$ represents instead non-zero amplitude (damaging) cycles. In the following, we shall use indifferently the notation $h_{\mathrm{LCC}}(u,\mathrm{v})$ or $h_{\mathrm{NB}}(u,\mathrm{v})$ for indicating the cycle distribution of the narrow-band approximation.

The distribution $h_{\mathrm{LCC}}(u,\mathrm{v})$ as given in Eq. (57) is quite general, and it is valid for both Gaussian and non-Gaussian symmetric loadings, if proper peak and valley distributions are used. In the case of Gaussian process, whose peak and valley distributions are known (see Eq. (14)), the density $h_{\mathrm{LCC}}(u,\mathrm{v})$ is a solution of the integral equation Eq. (25), confirming that the level-crossing count is a complete procedure (hence, $\nu_{\mathrm{a}} = \nu_{\mathrm{p}}$).

The joint distribution of amplitude and mean values associated to $h_{\mathrm{LCC}}(u,\mathrm{v})$, calculated as in Eq. (23), is:

$$p_{\mathrm{a,m}}^{\mathrm{LCC}}(s,m) = \tag{58}$$
$$= \begin{cases} \left[p_{\mathrm{p}}(s) - p_{\mathrm{v}}(s) \right] \delta(m) + p_{\mathrm{v}}(m)\, \delta(s) & \text{if } \; s+m>0 \\ p_{\mathrm{p}}(m)\, \delta(s) & \text{if } \; s+m \le 0 \end{cases}$$

and it is depicted in Figure 3(a). The marginal amplitude distribution is:

$$p_{\mathrm{a}}^{\mathrm{LCC}}(s) = [p_{\mathrm{p}}(s) - p_{\mathrm{v}}(s)] \; = \; \alpha_2 \, \frac{s}{\lambda_0} e^{-\frac{s^2}{2\lambda_0}} \tag{59}$$

and it is a Rayleigh density that, substituted into Eq. (31), with ν_{p} given by Eq. (6), gives the damage intensity $\overline{D}_{\mathrm{NB}}$ as in Eq. (38), i.e. the narrow-band approximation. This proves that, in the case of a Gaussian process, the joint density $h_{\mathrm{LCC}}(u,\mathrm{v})$ is the expression of the distribution of both the level-crossing count and that the upper bound of any crossing-consistent cycle count. In fact, integration of the $h_{\mathrm{LCC}}(u,\mathrm{v})$ distribution as in Eq. (26) gives its cumulative counting intensity:

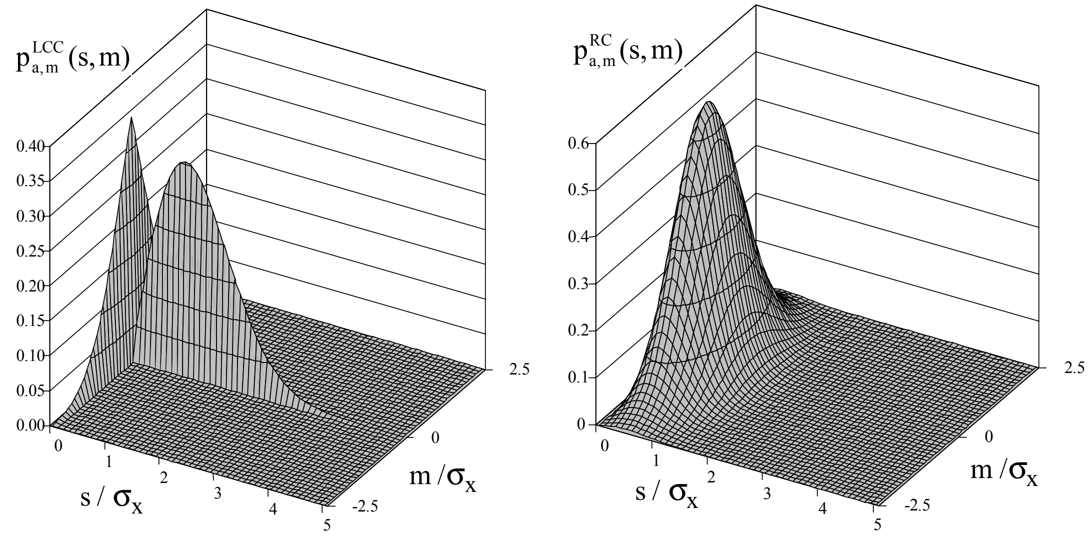
$$\mu_{\mathrm{LCC}}(u,\mathrm{v}) = \mu^{+}(u)\, \mathbf{I}\left(u+\mathrm{v} \ge 0\right) \; + \; \mu^{+}(\mathrm{v})\, \mathbf{I}\left(u+\mathrm{v} \le 0\right) \tag{60}$$

defined by means of an indicator function: $\mathbf{I}(x)=1$ if $x \ge 0$, elsewhere $\mathbf{I}(x)=0$. Equation above evidently equals the $\mu^{+}(u,\mathrm{v})$ distribution given in Eq. (19), which also shows that the level-crossing is a crossing-consistent count.

For the lower bound of the rainflow damage (i.e. the range-count damage), no exact analytical expression is known at present, thus we can adopt the approximate result proposed in [Madsen et al. 1986]:

$$\overline{D}_{\mathrm{RC}} \cong \nu_{\mathrm{P}}\, C\left(\sqrt{2\lambda_0}\,\alpha_2\right)^k \Gamma\left(1+\frac{k}{2}\right) = \overline{D}_{\mathrm{NB}}\, \alpha_2^{k-1} \tag{61}$$

Formula above has been obtained by studying the double envelope of a random process and by approximating the result of the range count by means of the amplitude of the envelope process (further details on differences between this definition and the actual range-count are given in [Madsen at. al. 1986]).



(a) (b)

Figure 3: Probability density distributions of counted cycles ($p^{\mathrm{LCC}}_{\mathrm{a,m}}$ representation is qualitative since Dirac delta function is not graphicable).

It is noteworthy that we also know explicitly the underlying cycle distribution $h_{\mathrm{RC}}(u,\mathrm{v})$, which gives the approximate range-count damage, $\overline{D}_{\mathrm{RC}}$, under the assumption of the linear damage accumulation rule. Precisely, let us consider the following particular joint density proposed in [Tovo 2002]:

$$h_{\mathrm{RC}}(u,\mathrm{v}) = \frac{1}{2\sqrt{2\pi}\lambda_0\,\alpha_2^2}\, e^{-\frac{u^2+\mathrm{v}^2}{4\lambda_0(1-\alpha_2^2)}}\, e^{-\frac{(u-\mathrm{v})^2}{4\lambda_0(1-\alpha_2^2)}\frac{1-2\alpha_2^2}{2\alpha_2^2}} \left[\frac{u-\mathrm{v}}{\sqrt{4\lambda_0(1-\alpha_2^2)}}\right] \tag{62}$$

The corresponding joint distribution of amplitudes and mean values, $p^{\mathrm{RC}}_{\mathrm{a,m}}(s,m)$, and the marginal amplitude distribution, $p^{\mathrm{RC}}_{\mathrm{a}}(s)$, are computed by applying Eqs. (23) and (24), respectively:

$$p^{\mathrm{RC}}_{\mathrm{a,m}}(s,m) = \frac{1}{\sqrt{2\pi\lambda_0(1-\alpha_2^2)}}\, e^{-\frac{m^2}{2\lambda_0(1-\alpha_2^2)}} \cdot \frac{s}{\lambda_0\,\alpha_2^2}\, e^{-\frac{s^2}{2\alpha_2^2\lambda_0}} \tag{63}$$

$$p^{\mathrm{RC}}_{\mathrm{a}}(s) = \frac{s}{\lambda_0\,\alpha_2^2}\, e^{-\frac{s^2}{2\alpha_2^2\lambda_0}} \tag{64}$$

The probability density $p^{\mathrm{RC}}_{\mathrm{a,m}}(s,m)$ is depicted in Figure 3(b).

The crucial point here is that cycle distribution $h_{\mathrm{RC}}(u,\mathrm{v})$ gives range-counting damage $\overline{D}_{\mathrm{RC}}$ under the Palmgren-Miner damage rule, i.e. when we substitute its related marginal amplitude distribution $p^{\mathrm{RC}}_{\mathrm{a}}(s)$ into Eq. (31). This means that the distribution yielded by Eqs. (62)-(64) causes the same damage, as that proposed by

Madsen et al. as an approximation of the range-count damage. Obviously, this does not mean that these densities are the cycle distribution resulting from the range-count, but it is reasonable that they do cause damage close to the lower bound of the rainflow counting damage.

Further researches have shown that $h_{\mathrm{RC}}(u,\mathrm{v})$ density coincides with an approximate function quantifying the transition probability between adjacent extremes in a Gaussian process, independently proposed by Sjöström and Kowalewski [Sjöström 1961, Kowalewski 1966] (Kowalewski's formula can be also found in [Bishop and Sherrat 1990]). In other words, Sjöström-Kowalewski' joint density, $h_{\mathrm{RC}}(u,\mathrm{v})$, can be viewed as an approximate distribution for cycles identified by the range-count, i.e. cycles constructed by pairing adjacent local extremes (e.g. a maximum and the following minimum). Other references related to this distribution are referred to Butler (1961) and to Cartwright and Longuet-Higgins (1956), as indicated in [Tunna 1986].

At this point, several properties of the $h_{\mathrm{RC}}(u,\mathrm{v})$ density are of interest. First of all, the count intensity calculated according to Eq. (26), after some manipulations, yields:

$$\mu_{\mathrm{RC}}(u,\mathrm{v}) = \nu^{+} \left\{ e^{-\frac{\mathrm{v}^2}{2\lambda_0}} \left[1-\Phi\left(\frac{u-\mathrm{v}\,(1-2\alpha_2^2)}{2\,\alpha_2\,\sqrt{\lambda_0\,(1-\alpha_2)}} \right) \right] + e^{-\frac{u^2}{2\lambda_0}}\, \Phi\left(\frac{\mathrm{v}-u\,(1-2\alpha_2^2)}{2\,\alpha_2\,\sqrt{\lambda_0\,(1-\alpha_2)}} \right) \right\} \tag{65}$$

where $\nu^{+} = \sqrt{\lambda_2/\lambda_0}\,/2\pi$ is the mean upcrossing rate. It is straightforward to prove that, for $u=\mathrm{v}$, previous expression converts into the upcrossing formula for a Gaussian process (Rice's formula), i.e.:

$$\begin{aligned} \mu_{\mathrm{RC}}(u,u) &= \nu^{+} \left\{ e^{-\frac{u^2}{2\lambda_0}} \left[1-\Phi\left(\frac{\alpha_2\,u}{\sqrt{\lambda_0\,(1-\alpha_2)}} \right) \right] + e^{-\frac{u^2}{2\lambda_0}}\, \Phi\left(\frac{\alpha_2\,u}{\sqrt{\lambda_0\,(1-\alpha_2)}} \right) \right\} \\ &= \nu^{+}\, e^{-\frac{u^2}{2\lambda_0}} = \mu(u) \end{aligned} \tag{66}$$

which confirms us that the counting method with $h_{\mathrm{RC}}(u,\mathrm{v})$ distribution is crossing-consistent. Additionally, for a Gaussian process, where peak and valley distribution are known (see Eq. (14)), it is possible to verify by integration that $h_{\mathrm{RC}}(u,\mathrm{v})$ density is a solution of Eq. (25); consequently, it represents the cycle distribution of a counting method which is also complete. Figure 4shows the level curves of the count intensities $\mu_{\mathrm{LCC}}(u,\,\mathrm{v})$ and $\mu_{\mathrm{RC}}(u,\,\mathrm{v})$ given in Eqs. (60) and (66) for a Gaussian process having $\lambda_0=\lambda_2=1$ and $\lambda_4=4$ ($\alpha_2=0.5$).

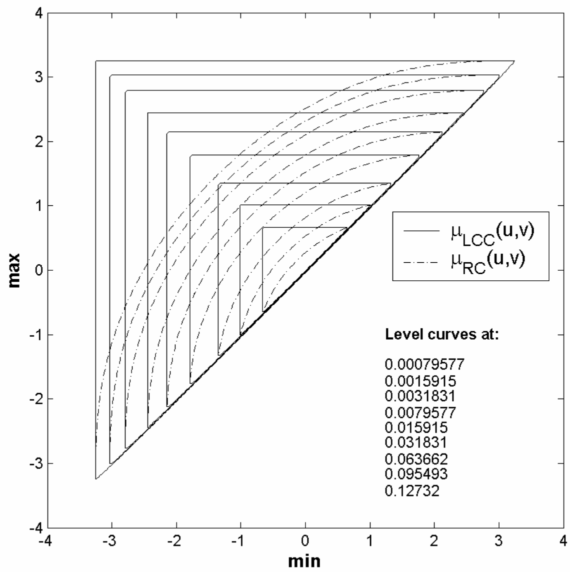


Figure 4: Comparison of $\mu_{\rm LCC}(u, \mathrm{v})$ and $\mu_{\rm RC}(u, \mathrm{v})$ count intensities for a Gaussian process with $\lambda_0 = \lambda_2 = 1$ and $\lambda_4 = 4$ ($\alpha_2 = 0.5$).

Secondly, careful consideration of Eq. (63) reveals that $p_{\rm a,m}^{\rm RC}(s,m)$ density actually represents independently distributed amplitude and mean value random variables, the former having a Rayleigh and the latter a Gaussian probability density function, with variance $\sigma_s^2 = \sqrt{\lambda_0\,(1-\alpha_2^2)}$ and $\sigma_m^2 = \lambda_0\,\alpha_2^2$, respectively. This can be condensed using the following symbolic notation:

$$p_{\rm a,m}^{\rm RC}(s,m) \overset{D}{=} \sqrt{\lambda_0\,(1-\alpha_2^2)}\;U \;\cdot\; \lambda_0^{1/2}\alpha_2\;R \tag{67}$$

where U is a standard normal variable and R a standard Rayleigh variable (U independent of R), and where $\overset{D}{=}$ denotes that two variables have same distribution. Note that $\sigma_s^2 + \sigma_m^2 = 1$. More specifically, it can be rigorously proved that, in a Gaussian process, exact independence between amplitudes and mean values of range-counted cycles would imply that amplitudes have a Rayleigh distribution and that mean values are Gaussian [Lindgren and Broberg 2004].

Some additional comments are now of interest. Systematic analysis of results from extensive numerical simulations has confirmed the correctness of the hypothesis of a Gaussian density (obviously with a zero mean value) for the distribution of mean values of range-counted cycles. At the same time, the hypothesis of a Rayleigh distribution for amplitudes has been found correct only for a restrict class of processes, namely for broad-band processes whose derivative is narrow-band ($\beta_1 \approx 1$; $\beta_2 \approx 1$). In both cases, the variance of the distribution is calculated as a polynomial, depending on α_1 (or q_X) and α_2 variables [Petrucci and Zuccarello 1999].

Consequently, for Gaussian broad-band processes with a narrow-band derivative, reasonably accurate fatigue life prediction are made assuming the hypothesis of independently distributed amplitudes and mean values (errors are less than 10 per cent). In the general case (e.g. for whatever broad-band process), the hypothesis of independence (on which the Sjöström-Kowalewski' density is based) leads generally to large errors [Petrucci and Zuccarello 1999].

For example, numerical simulations have shown that cycle means are almost independent of cycle amplitude only in a spectral density with a rectangular shape [Lindgren and Broberg 2004]. Anyway, we must always bear in mind that the independence assumption never holds for any Gaussian process with non-zero bandwidth, as demonstrated in [Lindgren 1970].

Previous discussion has evidenced that the distribution of range-counted cycle is strictly related to transition probability between adjacent extremes. This fact enables us for example to construct a Markov matrix after suitable normalisation of $h_{\mathrm{RC}}(u, \mathrm{v})$ density. Otherwise, the transition matrix can be estimated directly from measured or simulated data, or alternatively by numerical procedures. Two examples are mentioned here. The first regards a numerical-based approach for determining the one-step transition matrix (and thus the max-min cycle distribution) proposed in [Krenk and Gluver 1989]; the interesting fact is that transition probability of small ranges evidences a dependence on β_2 parameter, i.e. the irregularity factor of the derivative process. The second is a complete numerically-based method (available in WAFO toolbox) developed for the determination of the range-count cycle distribution; the method, although numerical, is exact in the sense that distribution becomes asymptotically exact when the integration grid increases [Lindgren and Broberg 2004].

We turn now to the evaluation of the rainflow fatigue damage. As stated by Eq. (56), the rainflow damage is always placed in-between previously defined bounds, namely $\overline{D}_{\mathrm{NB}}$ and $\overline{D}_{\mathrm{RC}}$, which for a given process (i.e. for a given spectral density) are fixed quantities, see Eqs. (38) and (61).

Thus, the problem of finding the rainflow damage intensity $\overline{D}_{\mathrm{RFC}}$ becomes the problem of finding the proper intermediate point between these bounds. Precisely, we suggest adopting a linear combination:

$$\overline{D}_{\mathrm{RFC}} = b\, \overline{D}_{\mathrm{NB}} + (1-b)\, \overline{D}_{\mathrm{RC}} \tag{68}$$

in which the b weighting factor depends on the spectral density of the process.

On the basis of Eq. (56), we can expect that a relation similar to Eq. (68) is also true for cycle distributions; therefore, we can estimate the distribution of rainflow counted cycles, $h_{\mathrm{RFC}}(u, \mathrm{v})$ (or equivalently its related cumulative distributions, $H_{\mathrm{RFC}}(u, \mathrm{v})$ or $\mu_{\mathrm{RFC}}(u, \mathrm{v})$) by using an analogous linear combination. For example, the rainflow joint density function is estimated as:

$$h_{\mathrm{RFC}}(u, \mathrm{v}) = b\, h_{\mathrm{LCC}}(u, \mathrm{v}) + (1-b)\, h_{\mathrm{RC}}(u, \mathrm{v}) \tag{69}$$

being $h_{\mathrm{LCC}}(u, \mathrm{v})$ and $h_{\mathrm{RC}}(u, \mathrm{v})$ the cycle distributions for level-crossing (i.e. the narrow-band approximation) and range counts, introduced in Eqs. (57) and (62). Since the distribution of range-counted cycles is computed by an approximate formula, the Eq. (69) holds only in a first approximation sense. In Figure 5 we show the rainflow joint density function, $h_{\mathrm{RFC}}(u, \mathrm{v})$, calculated as in Eq. (69), for a Gaussian process having variance $\sigma_X^2 = 1$.

Similarly, we can estimate the probability density of rainflow cycles as:

$$p_{\mathrm{a}}^{\mathrm{RFC}}(s) = b\, p_{\mathrm{a}}^{\mathrm{LCC}}(s) + (1-b)\, p_{\mathrm{a}}^{\mathrm{RC}}(s) \tag{70}$$

being $p_{\mathrm{a}}^{\mathrm{LCC}}(s)$ and $p_{\mathrm{a}}^{\mathrm{RC}}(s)$ the amplitude probability densities as given in Eqs. (59) and (64). The advantage of formula (69) is that it allows estimate the rainflow cycle distribution in terms of peak and valley levels, and this is the fundamental point for subsequent application of the method to non-Gaussian cases.

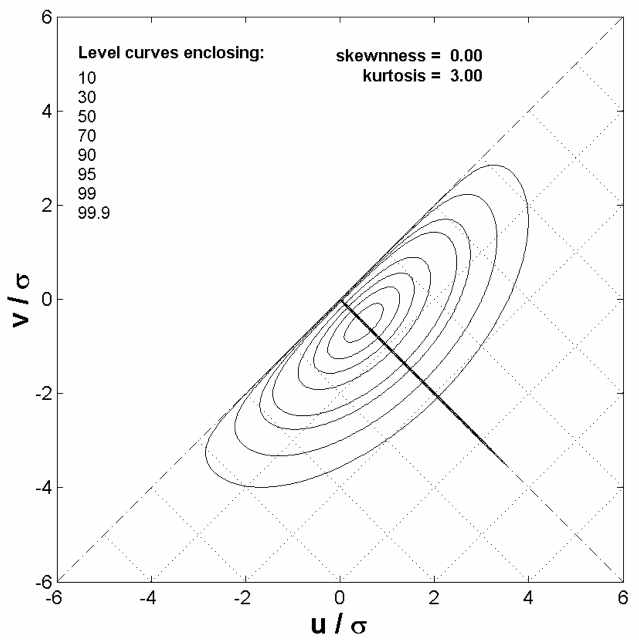


Figure 5: Rainflow cycle distribution $h_{\text{RFC}}(u, \text{v})$ as a function of peak u and valley v levels. Representation of Dirac delta function is qualitative.

Based on Eq. (61), the expression in Eq. (68) can be also written as:

$$\overline{D}_{\text{RFC}} = \left[\, b + (1-b)\,\alpha_2^k \,\right] \overline{D}_{\text{NB}} = \lambda_{\text{BT}}\ \overline{D}_{\text{NB}} \tag{71}$$

in which λ_{BT}, in analogy with Eq. (39), can be interpreted as a correction of the narrow-band approximation; however, the main difference here in respect to Wirsching and Light method, is that a more complete theoretical framework is behind the definition of λ_{BT} index.

Until now, no exhaustive theoretical information concerning b parameter and its dependence on process spectral density is available. However, some general b properties can be mentioned here. In a narrow-band process, $\alpha_2 = 1$ and then $\overline{D}_{\text{RFC}}$ equals $\overline{D}_{\text{NB}}$, whatever value b may have, which seems correct. Furthermore, when $k = 1$, Eq. (68) predicts $\overline{D}_{\text{RFC}} = \overline{D}_{\text{NB}}$, which is true [Rychlik 1993a, Lutes at al. 1984]. Finally, Eq. (68) is also applicable in the case of an irregular processes having $\alpha_2 = 0$, where it reduces to $\overline{D}_{\text{RFC}} = b\ \overline{D}_{\text{NB}}$.

More specifically, since we don't know the exact correlation relating b to spectral parameters, we must rely only on approximate formulas based on reasonable assumptions and then calibrated on numerical simulations; for example, the following formula [Tovo 2002]:

$$b_{\text{app}}(\alpha_1, \alpha_2) = \min\left\{ \frac{\alpha_1 - \alpha_2}{1-\alpha_1},\ 1 \right\} \tag{72}$$

implicitly assumes that the rainflow damage depends on just four spectral moments (i.e. λ_0, λ_1, λ_2 and λ_4) through α_1 and α_2 bandwidth parameters (compared to other methods, once again a further dependence on λ_1 moment is introduced, as in Dirlik's approach).

In the next section, based on numerically simulated results, we shall revise previous expression, giving a modified improved approximation still involving only α_1 and α_2 parameters. As discussed above, yet, a more complex dependence including other spectral parameters (as bandwidth parameters relative to the derivative process $\dot{X}(t)$) could exists.

5.3.4. Approximate method based on $\alpha_{0.75}$ bandwidth parameter

As observed in previous sections, a possible correlation of rainflow fatigue damage on some particular bandwidth parameters has been investigated, and a dependence on $\alpha_{0.75} = \lambda_{0.75} \big/ \sqrt{\lambda_0 \, \lambda_{1.5}}$ has been suggested in [Lutes et al. 1984].

On a pure empirical basis, one can argue that the correction factor $\lambda = \overline{D}_{\mathrm{RFC}} \big/ \overline{D}_{\mathrm{NB}}$ is a function of $\alpha_{0.75}$ bandwidth parameter, and that it is independent of S-N slope k. We suggest the following simple formulation:

$$\overline{D}_{\mathrm{RFC}} = \lambda_{\mathrm{BT}} \, \overline{D}_{\mathrm{NB}} \qquad \lambda_{\mathrm{BT}} = \alpha_{0.75}^2 \tag{73}$$

This expression, even if approximate and lacking of any theoretical motivation, has been proved to agree fairly well with data from simulation (see later on) and can be taken as a first approximation of the rainflow damage.

6. NUMERICAL SIMULATIONS

The main goal of numerical simulations is to validate the accuracy of several spectral methods in estimating, in Gaussian broad-band processes, both the rainflow cycles distribution and the rainflow fatigue damage under the Palmgren-Miner law. Secondly, the aim is also to investigate the true set of spectral parameters actually defining the correlation existing between the spectral density and the rainflow cycle distribution. Results of this section will be mainly focused on the properties of *b* index, defined by Eq. (68).

As discussed before, the true set of spectral moments controlling the rainflow cycle distribution is actually not known, and we can only rely on reasonable hypotheses based on simulations.

The fundamental assumption here is that distribution of rainflow cycles mainly depends on α_1 and α_2 bandwidth parameters; thus, the need of different spectral densities having the same α_1 and α_2 pair (or, alternatively, different bandwidth indexes from the same spectrum geometry) is assumed as the guideline of our simulations.

Several stochastic processes were numerically simulated by assuming different shapes of the spectral density; namely, various one-sided spectral densities, $W_X(\omega)$ were considered, having simple geometries like that depicted in Figure 6, e.g. constant, linear, double symmetrical or anti-symmetric parabolic shape.

All spectra have the same variance (i.e. λ_0 moment) and common values for ω_1 and ω_3 frequencies (see Table 1), whereas ω_2 can move arbitrarily inside them. For a given spectrum, and for a given ω_2 value, a well-defined set of α_1 and α_2 bandwidth parameters is univocally established by selecting a proper (h_1, h_2) pair.

Table 1: Common parameters in numerically simulated processes.

$\omega_1 = 2\pi$	$\omega_3 = 10^5 \, \omega_1$	$\lambda_0 = \sigma_X^2 = 10^4$

The complete set of numerical simulations focused on α_2 values of 0.1, 0.3, 0.5 and 0.7, with α_1 taking specified increasing values between 0.1 and 0.9. For a given couple of these two indexes, the choice of different spectral density shapes make possible for β_1 parameter (i.e. the irregularity factor of the derivative process $X(t)$) to range from 0.500 to 0.950. Figure 7 gives some examples of parts of simulated processes with different combinations of bandwidth parameters.

In each simulation test (i.e. given a spectral density) a time history is generated in time domain and then the traditional time-domain analysis is performed: first, cycles are counted by means of the rainflow count, then fatigue damage is computed under the linear damage accumulation law. For fatigue damage computation, we assume two values for the slope *k* of the S-N relation (i.e. $k = 3$ and $k = 5$) and a reference strength *C* equal to unity.

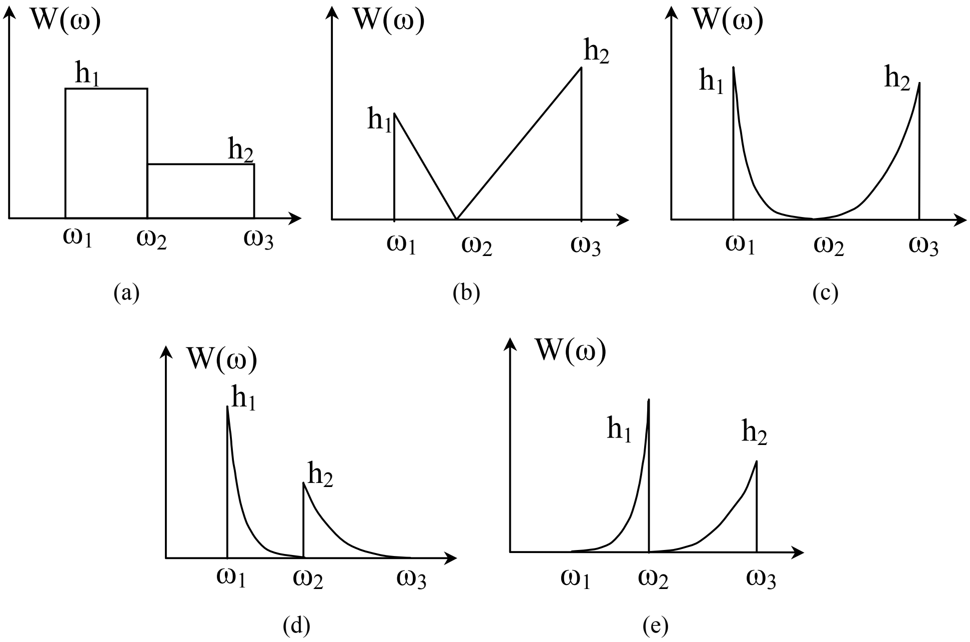


Figure 6: Spectral densities used in numerical simulations.

In the following, results made in time-domain (i.e. cycle distribution and fatigue damage) will be compared with predictions made in the frequency-domain by some spectral methods reviewed in previous sections.

Precisely, four methods have been considered, namely the Wirsching and Light' correction formula, Eqs. (39)-(41), Zhao and Baker method, Eq. (54), Dirlik's empirical method, Eq. (46), and the new method described in the last section and synthesized in Eqs. (68) and (72).

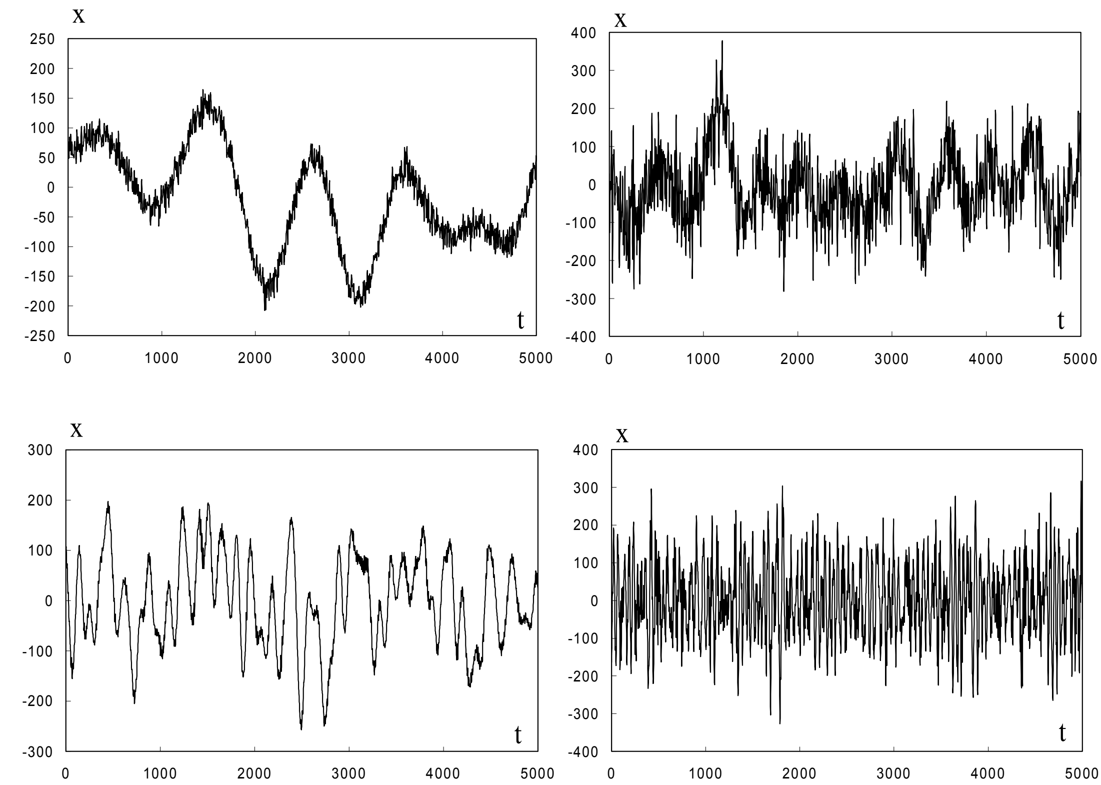
Although the approximate formula for b weighting factor given in Eq. (72) has been shown to be fairly accurate [Tovo 2002], our intention is to find an improved version of that formula. In order to do this, we have to understand the intrinsic relationship relating *b* coefficient to α_1 and α_2 bandwidth parameters, therefore we need to express Eq. (72) as a function of quantities resulting in simulations.

Time-domain calculation on a given simulated time history provides a fatigue damage value, say $\hat{D}_{\mathrm{RFC}}$, which can be taken as an estimate of the expected rainflow damage. Thus, inverting Eq. (68) and substituting this damage value gives an estimate of *b* factor as:

$$\hat{b} = \frac{\hat{D}_{\mathrm{RFC}} - \overline{D}_{\mathrm{RC}}}{\overline{D}_{\mathrm{NB}} - \overline{D}_{\mathrm{RC}}} \tag{74}$$

being $\overline{D}_{\mathrm{NB}}$ and $\overline{D}_{\mathrm{RC}}$ the damage intensities calculated by Eqs. (38) and (61) which are only functions of the process spectral density.

Repeating such calculation for all simulation tests (i.e. for all spectral densities) provide a set of $\hat{b}$ values for different α_1 and α_2 pairs; all such simulation results are shown as marked points in Figure 8.



(a) (b)

Figure 7: Time histories having various α_1 values and: (a) $\alpha_2 = 0.1$; (b) $\alpha_2 = 0.5$.

Since, as stated by Eq. (68), bandwidth parameters control rainflow damage through *b* coefficient, Figure 8 clearly evidences a strong dependence of damage on α_1 index, even for a constant α_2 value. Consequently, we may presume to have to consider also α_1 index in damage estimation formulas (as suggested by Dirlik's approximate expression and the new method). In any case, the relative spread of simulated results observed in Figure 8 (associated to different β_1 values) cannot be completely disregarded, meaning that a slight variation on damage, also for constant values of α_1 and α_2 parameters, can sometimes be observed.

Anyway, even if conscious of a possible additional relationship attributable to β_1 index, is our opinion to focus to a functional relationship still involving α_1 and α_2 parameters (as already proposed in Eq. (68)); a closed-form of such relation is given in analytical form by the following expression:

$$b_{\text{app}} = \frac{(\alpha_1 - \alpha_2)\left[1.112\left(1 + \alpha_1\alpha_2 - (\alpha_1 + \alpha_2)\right)e^{2.11\alpha_2} + (\alpha_1 - \alpha_2)\right]}{(\alpha_2 - 1)^2} \tag{75}$$

which, added in Figure 8 as a continuous line, is in fairly good agreement with all numerical results.

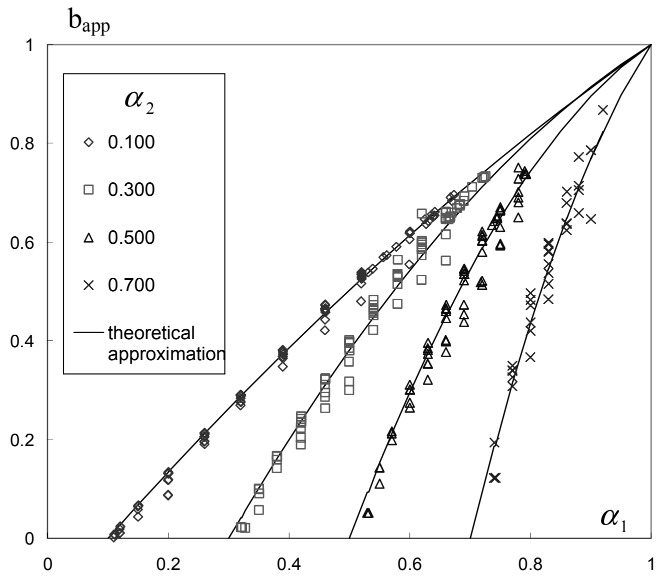


Figure 8: Comparison between $\hat{b}$ values in numerical simulations and approximate analytical prediction, Eq.(**3.76**)

Nevertheless, a possible dependence of rainflow damage on $\alpha_{0.75}$ bandwidth parameter may exist. In Figure 9 we show the correlation existing between the expression $\lambda_{BT} = \alpha_{0.75}^2$ (continuous line) and results from simulations (dots).

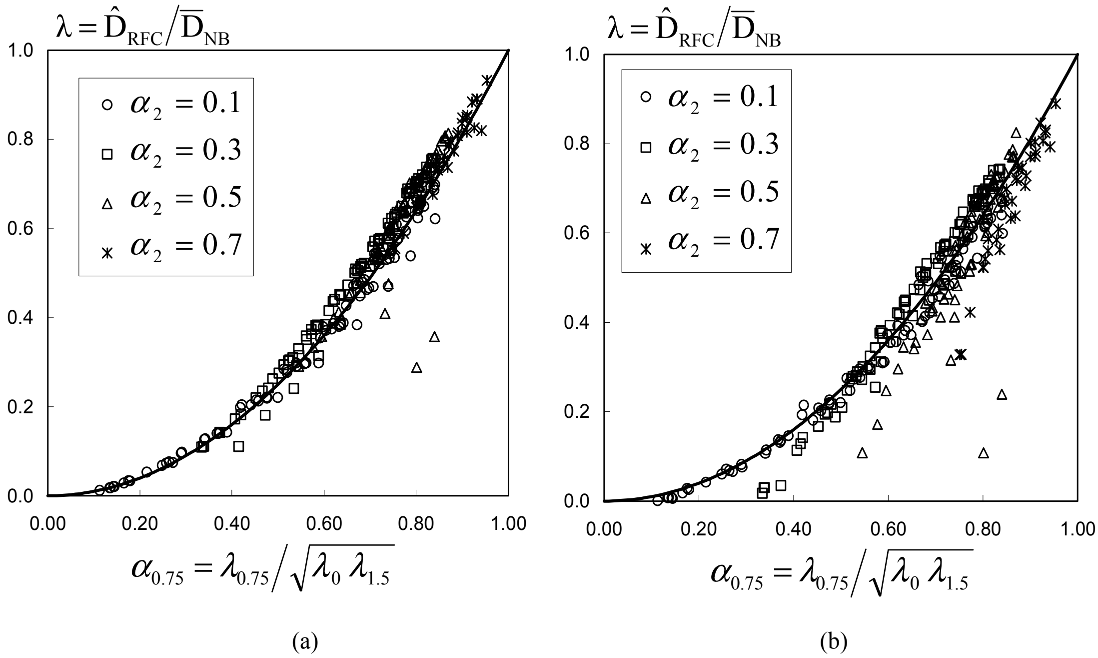


Figure 9: Relationship between $\alpha_{0.75}$ and λ_{BT} correction factor. The expression $\lambda_{BT} = \alpha_{0.75}^2$ (continuous line) is compared with simulations (dots). S-N slope (a) $k = 3$ and (b) $k = 5$.

As can be seen, all data are very close to the proposed expression, except for some points, independently of the S-N slope k.

At this point, we can compare damage values, the b coefficient in Eq. (68) being computed according to the approximate expression given in Eq. (75).

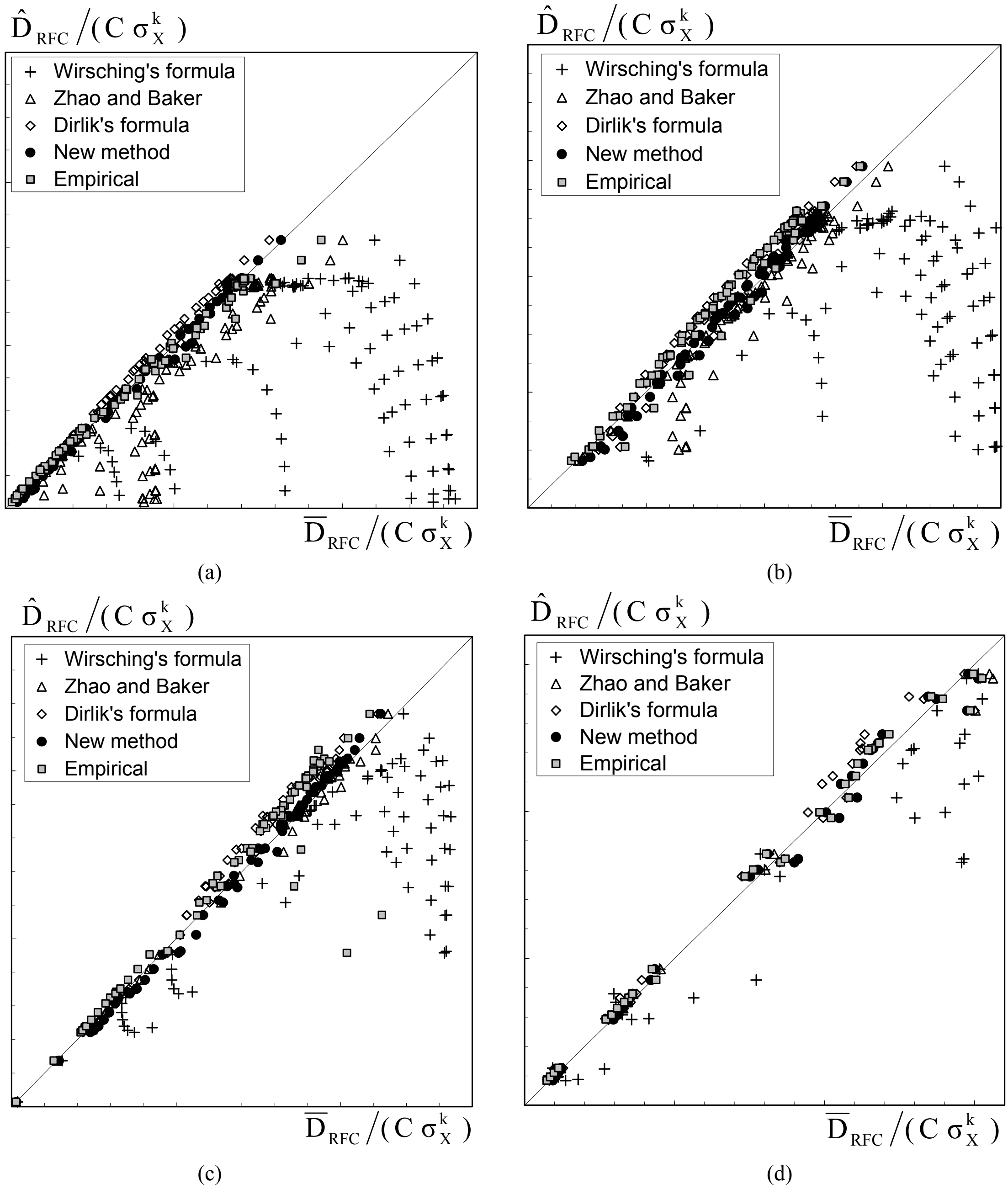


Figure 10: Comparison of damages for processes having different irregularity factors. Slope $k = 3$. (a) $\alpha_2 = 0.1$; (b) $\alpha_2 = 0.3$; (c) $\alpha_2 = 0.5$; (d) $\alpha_2 = 0.7$.

Results concerning rainflow fatigue damage from Gaussian simulation for each of the spectral densities types studied, having α_2 equal to 0.1, 0.3, 0.5 and 0.7, are compared in Figure 10 and Figure 11.

In each figure, the abscissa of the data points is the expected rainflow damage intensity as estimated by a spectral method, and the ordinate is the damage intensity as calculated by the rainflow analysis of the simulated data.

Thus, perfect correspondence between a spectral method and a simulation is indicated by data lying on the straight line, representing the bisector of the damage plane. Any deviations of the data from this line indicate inaccuracies in the spectral methods. Damage values are all normalised to the fatigue strength C and to σ_X^k, being constant for all spectral densities analysed.

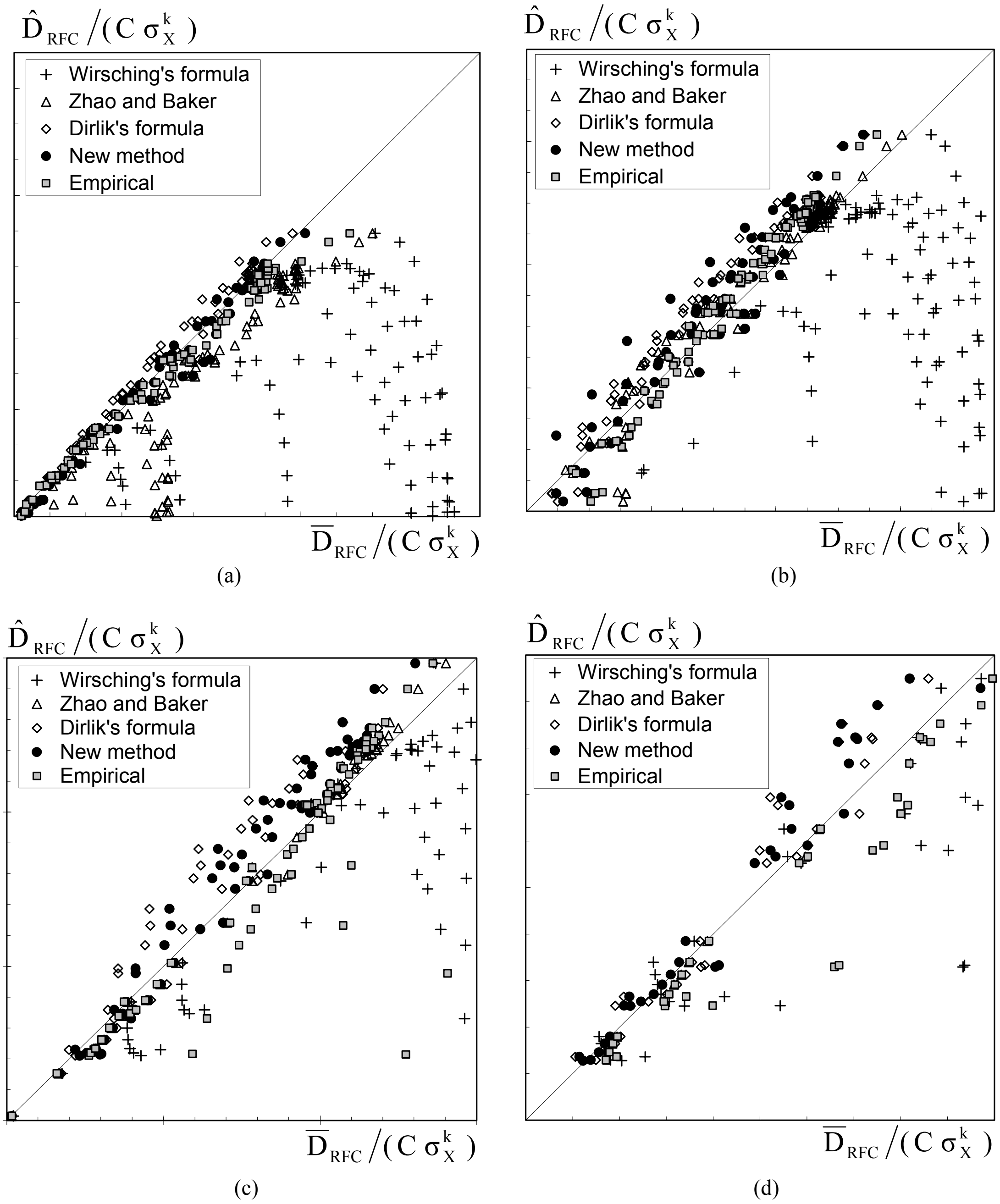


Figure 11: Comparison of damages for processes having different irregularity factors. Slope $k=5$. (a) $\alpha_2=0.1$; (b) $\alpha_2=0.3$; (c) $\alpha_2=0.5$; (d) $\alpha_2=0.7$.

Results presented in all figures evidence how Wirsching and Light' approach, even if quite simple if compared to other existing methods, tends to systematically give too conservative predictions in respect to simulated damage. The accuracy of the prediction increases when α_2 tends to unity, i.e. when the process tends to be narrow-band, which seems plausible.

On the contrary, better agreement with results from simulations seems to be associated with other spectral method. Namely, Dirlik's, Zhao and Baker' and the new method show better agreement between the estimated rainflow damage and the damage computed from simulations, at least for slope k equal to 3 (see Figure 10).

For what concerns the Zhao and Baker' technique, our calculations have evidenced that the simplest formulation, using Eqs. (49) and (50), is fairly inaccurate. In addition, for all spectra having $\alpha_2=0.1$ it incorrectly predicts a weight $w>1$, which is not correct. On the contrary, the alternative improved version, Eqs. (51) and (52) (that is presented in all figures) show better results.

For the case of a S-N slope k equal to 5 a greater scatter amongst all results is observed (see Figure 11). In particular, it has to be underlined that for low values of the irregularity index α_2, the spread between the two bounds $\overline{D}_{\mathrm{NB}}$ and $\overline{D}_{\mathrm{RC}}$ of the rainflow damage defined in Eq. (56) is too high (ratio $\overline{D}_{\mathrm{RC}} / \overline{D}_{\mathrm{NB}}$ is about α_2^{k-1}), and this information is not sufficiently accurate for reliable fatigue damage assessment.

In conclusion, we can affirm that the new method is as accurate as the Dirlik's method, which is recognised as the best predictor for rainflow damage. This, on the contrary, does not hold for Wirsching and Light' correction formula, which has shown great inaccuracy.

A further advantage of the new method and Dirlik's method is their capability to estimate also the rainflow cycle distribution; in Figure 12 we compare the expected loading spectra with a sample taken from numerical simulations. For the Dirlik's method, we make use of Eq. (47), whereas for the new method we compute the loading spectrum by substituting into Eq. (27) the estimated rainflow amplitude distribution $p_{\mathrm{a}}^{\mathrm{RFC}}(s)$ computed by Eq (70), which is equivalent to directly computing the following linear combination:

$$F^{\mathrm{RFC}}(s) = b\, F^{\mathrm{LCC}}(s) + (1-b)\, F^{\mathrm{RC}}(s) \tag{76}$$

between the fatigue loading spectra relative to the level-crossing count, $F^{\mathrm{LCC}}(s)$, and the range-count, and $F^{\mathrm{RC}}(s)$, calculated as:

$$\begin{aligned} F^{\mathrm{LCC}}(s) &= \alpha_2\, e^{-\frac{s^2}{2\lambda_0}} \\ F^{\mathrm{RC}}(s) &= e^{-\frac{s^2}{2\alpha_2^2 \lambda_0}} \end{aligned} \tag{77}$$

From Figure 12, it can be seen how simulated rainflow spectra are well predicted by both methods. However, it is worth noting that the new method differs significantly from Dirlik's formulation in the fact that it possesses a sound theoretical background, whereas Dirlik's approach has not theoretical framework.

We conclude this section with other results, which aim to evaluate the accuracy of the approximate formula, Eq. (61), used for computing the expected range-count damage $\overline{D}_{\mathrm{RC}}$.

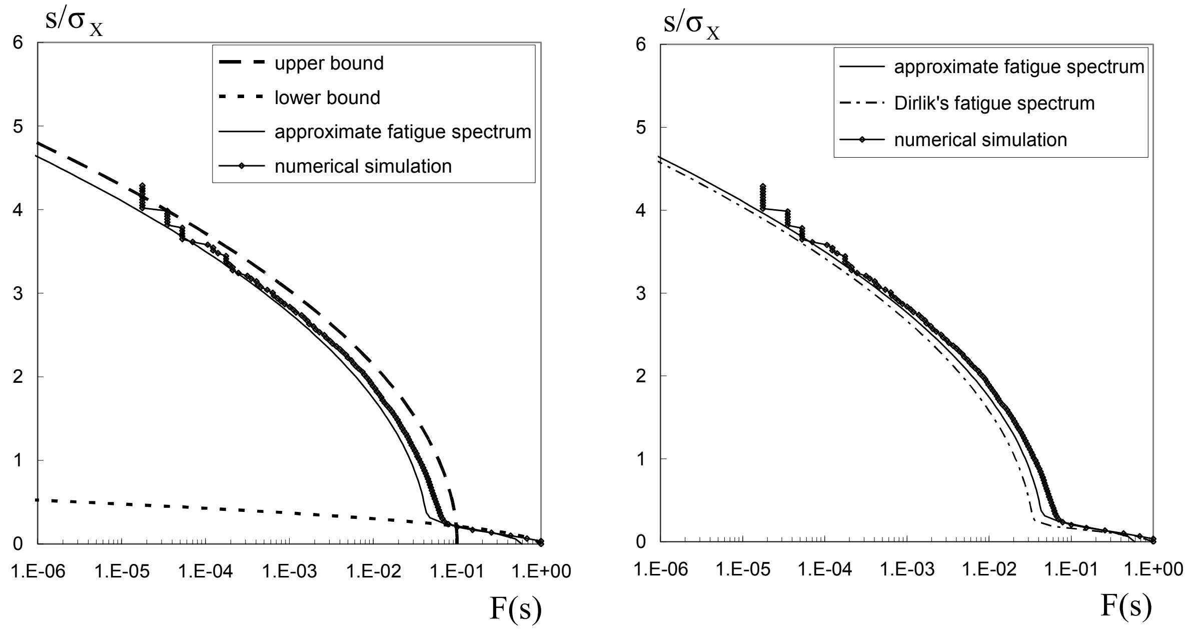


Figure 12: Comparison between numerical fatigue spectra and theoretical estimations (for $\alpha_2 = 0.1$). New method (left); Dirlik's model (right).

Comparisons are made in terms or the expected damage per cycle $\overline{D}_{RC} / \nu_p$, which according on Eq. (61) only depends on the irregularity factor α_2. Therefore, for each value of the irregularity index, the theoretical formula predicts a constant value of the damage.

On the other hand, it can be observed that the range-count damage resulting fro simulations depends also on α_1, and in particular the difference between the observed and the estimated damage per cycle tends to increase with the difference $\Delta = \alpha_1 - \alpha_2$.

This confirms us that Eq. (61) is in fact an approximate formula; however, when it looses it accuracy, the b coefficients tends to unity, which makes the contribution of the $\overline{D}_{RC}$ damage in calculating the rainflow damage less important.

7. CONCLUSIONS

The rainflow cycle distribution and the fatigue damage under the linear damage rule in Gaussian broad-band random loadings have been analysed. The theory concerning random loadings is reviewed and several analytical formulas are presented, as the peak approximation, the narrow-band approximation, and other methods specifically developed for broad-band random processes (e.g. Wirsching and Light' correction formula, Dirlik's approximate rainflow amplitude density, Zhao and Baker' method).

Subsequently, by simple theoretical considerations we have presented the development of our new method, in which the rainflow damage is estimated as weighted linear combination from two damage values, each corresponding to the damage from the narrow-band approximation and from the range counting method, respectively. An approximate function of both bandwidth parameters α_1 and α_2 (i.e. still four spectral parameters) has been proposed for the weighting parameter, *b*.

In order to check the validity of previously reviewed estimation formulas, we have performed numerical simulations, comparing damage values calculated in time domain (i.e. with rainflow count and Palmgren-Miner law) with analytical expressions, depending on frequency-domain quantities.

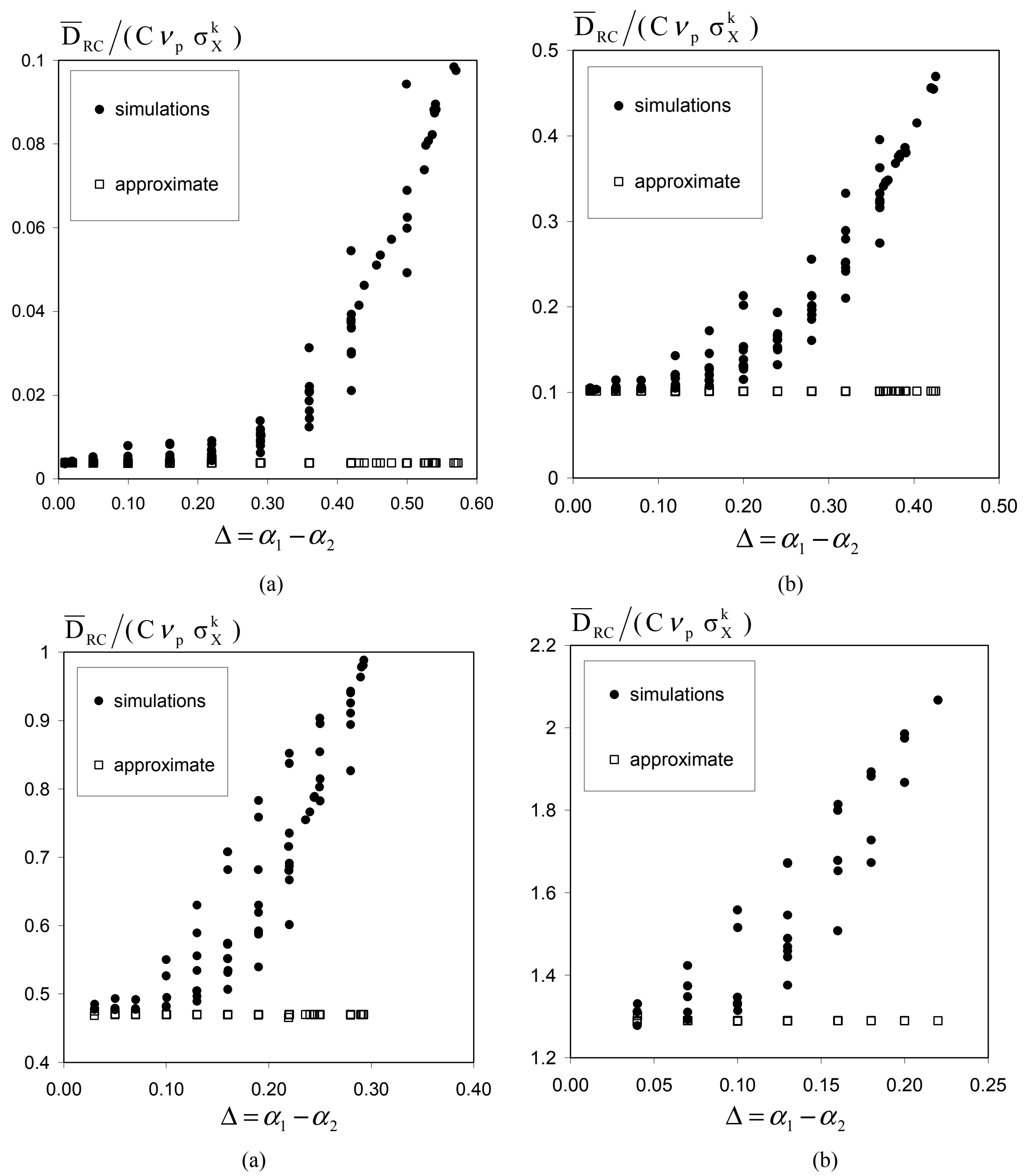


Figure 13: Range-count damage per cycle (slope $k = 3$) obtained from simulations and computed according to approximate formula proposed by Madsen et al. (a) $\alpha_2 = 0.1$; (b) $\alpha_2 = 0.3$; (c) $\alpha_2 = 0.5$; (d) $\alpha_2 = 0.7$.

First, we have considered spectral densities with simple shapes, so to get a systematic investigation of the dependence between spectral bandwidth parameters and rainflow fatigue damage in wide-band random processes. Then, a simple model travelling on a road has been considered: in this application, the effect produced on damage by varying one model parameter can be clearly evidenced.

Our work confirms that the so-called narrow-band approximation gives in wide band processes over conservative results (i.e. it estimates a damage greater that the true value); in fact it constitutes only an upper bound for expected rainflow damage; however, also a correction formula proposed by Wirsching and Light doesn't work very well.

On the other hand, the approximate expression proposed by Dirlik gives better results, even if no theoretical framework is given (i.e. it represents a completely approximate formula). So, we have shown how also our method is able to give good estimation results, even if it should be stressed the fact that the method itself is supported in parallel by a corresponding theoretical background. In addition, we wish also to emphasise the advantage of our method in respect to Dirlik's approach represented by the possibility to get the complete distribution of rainflow counted cycles in terms of amplitudes and mean values. With in minds these considerations, we are lead to conclude that our method seems a very promising approach in solving the rainflow damage estimation problem.

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